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ELECTROMAGNETIC FIELD THEORY
EXERCISES

PLEASE NOTE THAT THIS IS A

VERY PRELIMINARY DRAFT!

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Companion volume to

ELECTROMAGNETIC FIELD THEORY

by

Bo Thidé

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Exercises

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VERY preliminary draft!

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This is a companion volume to the book *Electromagnetic Field Theory* by Bo Thidé. The problems and their solutions were created by the co-authors who all have taught this course or its predecessor.

It should be noted that this is a *preliminary draft* version but it is being corrected and expanded with time.

Uppsala, Sweden   B. T.
December, 1999

Maxwell's Equations

1.1 Coverage

In this lesson we examine Maxwell's equations, the cornerstone of electrodynamics. We start by practising our math skill, refreshing our knowledge of vector analysis in vector form and in component form.

1.2 Formulae used

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \tag{1.1a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.1b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{1.1c}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{1.1d}$$

1.3 Solved examples

▷MACROSCOPIC MAXWELL EQUATIONS———————————————— EXAMPLE 1.1

The most fundamental form of Maxwell's equations is

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \tag{1.2a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.2b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{1.2c}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{1.2d}$$

sometimes known as the microscopic Maxwell equations or the Maxwell-Lorentz equations. In the presence of a medium, these equations are still true, but it may sometimes be convenient to separate the sources of the fields (the charge and current densities) into an induced part, due to the response of the medium to the electromagnetic fields, and an extraneous, due to "free" charges and currents not caused by the material properties. One then writes

$$\mathbf{j} = \mathbf{j}_{\text{ind}} + \mathbf{j}_{\text{ext}} \tag{1.3}$$
$$\rho = \rho_{\text{ind}} + \rho_{\text{ext}} \tag{1.4}$$

The electric and magnetic properties of the material are often described by the electric polarisation $\mathbf{P}$ (SI unit: C/m^2) and the magnetisation $\mathbf{M}$ (SI unit: A/m). In terms of these, the induced sources are described by

$$\mathbf{j}_{\text{ind}} = \partial \mathbf{P}/\partial t + \nabla \times \mathbf{M} \tag{1.5}$$
$$\rho_{\text{ind}} = -\nabla \cdot \mathbf{P} \tag{1.6}$$

To fully describe a certain situation, one also needs constitutive relations telling how $\mathbf{P}$ and $\mathbf{M}$ depends on $\mathbf{E}$ and $\mathbf{B}$. These are generally empirical relations, different for different media.

Show that by introducing the fields

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{1.7}$$
$$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} \tag{1.8}$$

the two Maxwell equations containing source terms (1.2a) and (**??**) reduce to

$$\nabla \cdot \mathbf{D} = \rho_{\text{ext}} \tag{1.9}$$
$$\nabla \times \mathbf{H} = \mathbf{j}_{\text{ext}} + \frac{\partial}{\partial t}\mathbf{D} \tag{1.10}$$
$$\tag{1.11}$$

known as the macroscopic Maxwell equations.

Solution

If we insert

$$\mathbf{j} = \mathbf{j}_{\text{ind}} + \mathbf{j}_{\text{ext}} \tag{1.12}$$
$$\rho = \rho_{\text{ind}} + \rho_{\text{ext}} \tag{1.13}$$

and

$$\mathbf{j}_{\text{ind}} = \frac{\partial}{\partial t}\mathbf{P} + \boldsymbol{\nabla} \times \mathbf{M} \tag{1.14}$$

$$\rho_{\text{ind}} = -\boldsymbol{\nabla} \cdot \mathbf{P} \tag{1.15}$$

into

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{1.16}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\varepsilon_0 \tag{1.17}$$

$$\tag{1.18}$$

we get

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{j}_{\text{ext}} + \frac{\partial}{\partial t}\mathbf{P} + \boldsymbol{\nabla} \times \mathbf{M}) + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{1.19}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon_0}(\rho_{\text{ext}} - \boldsymbol{\nabla} \cdot \mathbf{P}) \tag{1.20}$$

which can be rewritten as

$$\boldsymbol{\nabla} \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{j}_{\text{ext}} + \frac{\partial}{\partial t}(\mathbf{P} + \varepsilon_0\mathbf{E}) \tag{1.21}$$

$$\boldsymbol{\nabla} \cdot (\varepsilon_0\mathbf{E} + \mathbf{P}) = \rho_{\text{ext}} \tag{1.22}$$

Now by introducing the $\mathbf{D}$ and the $\mathbf{H}$ fields such that

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \tag{1.23}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{1.24}$$

the Maxwell equations become

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j}_{\text{ext}} + \frac{\partial}{\partial t}\mathbf{D} \tag{1.25}$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_{\text{ext}} \tag{1.26}$$

QED∎

The reason these equations are known as "macroscopic" are that the material properties described by $\mathbf{P}$ and $\mathbf{M}$ generally are average quantities, not considering the atomic properties of matter. Thus $\mathbf{E}$ and $\mathbf{D}$ get the character of averages, not including details around single atoms etc. However, there is nothing in principle preventing us from using large-scale averages of $\mathbf{E}$ and $\mathbf{B}$, or even to use atomic-scale calculated $\mathbf{D}$ and $\mathbf{H}$ although this is a rather useless procedure, so the nomenclature "microscopic/macroscopic" is somewhat misleading. The inherent difference lies in how a material is treated, not in the spatial scales.

—End of example 1.1◁

EXAMPLE 1.2   ▷MAXWELL'S EQUATIONS IN COMPONENT FORM─────────────────────

Express Maxwell's equations in component form.

Solution

Maxwell's equations in vector form are written:

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \tag{1.27}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.28}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{1.29}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{1.30}$$

In these equations, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{j}$ are vectors, while ρ is a scalar. Even though all the equations contain vectors, only the latter pair are true vector equations in the sense that the equations themselves have several components.

When going to component notation, all scalar quantities are of course left as they are. Vector quantities, for example $\mathbf{E}$, can always be expanded as $\mathbf{E} = \sum_{j=1}^{3} E_j \hat{\mathbf{x}}_j = E_j \hat{\mathbf{x}}_j$, where the last step assumes Einstein's summation convention: if an index appears twice in the same term, it is to be summed over. Such an index is called a *summation index.* Indices which only appear once are known as *free indices,* and are not to be summed over. What symbol is used for a summation index is immaterial: it is always true that $a_i b_i = a_k b_k$, since both these expressions mean $a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}$. On the other hand, the expression $a_i = a_k$ is in general not true or even meaningful, unless $i = k$ or if $\mathbf{a}$ is the null vector.

The three E_j are the components of the vector $\mathbf{E}$ in the coordinate system set by the three unit vectors $\hat{\mathbf{x}}_j$. The E_j are real numbers, while the $\hat{\mathbf{x}}_j$ are vectors, i.e. geometrical objects. Remember that though they are real numbers, the E_j are *not* scalars.

Vector equations are transformed into component form by forming the scalar product of both sides with the same unit vector. Let us go into ridiculous detail in a very simple case:

$$\mathbf{G} = \mathbf{H} \tag{1.31}$$

$$\mathbf{G} \cdot \hat{\mathbf{x}}_k = \mathbf{H} \cdot \hat{\mathbf{x}}_k \tag{1.32}$$

$$(G_j \hat{\mathbf{x}}_j) \cdot \hat{\mathbf{x}}_k = (H_i \hat{\mathbf{x}}_i) \cdot \hat{\mathbf{x}}_k \tag{1.33}$$

$$G_j \delta_{jk} = H_i \delta_{ik} \tag{1.34}$$

$$G_k = H_k \tag{1.35}$$

This is of course unnecessarily tedious algebra for an obvious result, but by using this careful procedure, we are certain to get the correct answer: the free index in the resulting equation necessarily comes out the same on both sides. Even if one does not follow this complicated way always, one should to some extent at least *think* in those terms.

Nabla operations are translated into component form as follows:

$$\nabla\phi = \hat{x}_i\frac{\partial}{\partial x_i}\Phi \longrightarrow \frac{\partial\phi}{\partial x_i} \tag{1.36}$$

$$\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{\partial}{\partial x_i}V_i \longrightarrow \frac{\partial V_i}{\partial x_i} \tag{1.37}$$

$$\boldsymbol{\nabla}\times\mathbf{V} = \varepsilon_{ijk}\hat{x}_i\frac{\partial}{\partial x_j}V_k \longrightarrow \varepsilon_{ijk}\frac{\partial V_k}{\partial x_j} \tag{1.38}$$

where $\mathbf{V}$ is a vector field and ϕ is a scalar field.

Remember that in vector valued equations such as Ampère's and Faraday's laws, one must be careful to make sure that the free index on the left hand side of the equation is the same as the free index on the right hand side of the equation. As said above, an equation of the form $A_i = B_j$ is almost invariably in error!

With these things in mind we can now write Maxwell's equations as

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \longrightarrow \frac{\partial E_i}{\partial x_i} = \frac{\rho}{\varepsilon_0} \tag{1.39}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \longrightarrow \frac{\partial B_i}{\partial x_i} = 0 \tag{1.40}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \longrightarrow \varepsilon_{ijk}\frac{\partial E_k}{\partial x_j} = -\frac{\partial}{\partial t}B_i \tag{1.41}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \longrightarrow \varepsilon_{ijk}\frac{\partial B_k}{\partial x_j} = \mu_0 j_i + \frac{1}{c^2}\frac{\partial E_i}{\partial t} \tag{1.42}$$

─── END OF EXAMPLE 1.2 ◁

▷ THE CHARGE CONTINUITY EQUATION ──────────────────── EXAMPLE 1.3

Derive the continuity equation for charge density ρ from Maxwell's equations using (a) vector notation and (b) component notation. Compare the usefulness of the two systems of notations. Also, discuss the physical meaning of the charge continuity equation.

Solution

Vector notation In vector notation, a derivation of the continuity equation for charge looks like this:

Compute $\boldsymbol{\nabla}\cdot\frac{\partial}{\partial t}\mathbf{E}$ in two ways:

1. Apply $\frac{\partial}{\partial t}$ to Gauss's law:

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{E}) = \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\rho \tag{1.43}$$

2. Take the divergence of the Ampère-Maxwell law:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = \mu_0 \boldsymbol{\nabla} \cdot \mathbf{j} + \frac{1}{c^2} \boldsymbol{\nabla} \cdot \frac{\partial}{\partial t} \mathbf{E} \tag{1.44}$$

Use $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times) \equiv 0$ and $\mu_0 \varepsilon_0 c^2 = 1$:

$$\Longrightarrow \boldsymbol{\nabla} \cdot \frac{\partial}{\partial t} \mathbf{E} = -\frac{1}{\varepsilon_0} \boldsymbol{\nabla} \cdot \mathbf{j} \tag{1.45}$$

Comparison shows that

$$\frac{\partial}{\partial t} \rho + \boldsymbol{\nabla} \cdot \mathbf{j} = 0. \tag{1.46}$$

Component notation   In component notation, a derivation of the continuity equation for charge looks like this:

Compute $\frac{\partial}{\partial x_i} \frac{\partial}{\partial t} E_i$ in two ways:

1. Take $\frac{\partial}{\partial t}$ of Gauss's law:

$$\frac{\partial}{\partial t} \frac{\partial E_i}{\partial x_i} = \frac{1}{\varepsilon_0} \frac{\partial}{\partial t} \rho \tag{1.47}$$

2. Take the divergence of the Ampère-Maxwell law:

$$\frac{\partial}{\partial x_i} \left(\varepsilon_{ijk} \frac{\partial B_k}{\partial x_j} \right) = \mu_0 \frac{\partial}{\partial x_i} j_i + \frac{1}{c^2} \frac{\partial}{\partial x_i} \frac{\partial E_i}{\partial t} \tag{1.48}$$

Use that the relation $\varepsilon_{ijk} A_i A_j \equiv 0$ is valid also if $A_i = \frac{\partial}{\partial x_i}$, and that $\mu_0 \varepsilon_0 c^2 = 1$:

$$\longrightarrow \frac{\partial}{\partial t} \frac{\partial E_i}{\partial x_i} = -\frac{1}{\varepsilon_0} \frac{\partial}{\partial x_i} j_i \tag{1.49}$$

Comparison shows that

$$\frac{\partial}{\partial t} \rho + \frac{\partial j_i}{\partial x_i} = 0. \tag{1.50}$$

Comparing the two notation systems We notice a few points in the derivations above:

- It is sometimes difficult to see what one is calculating in the component system. The vector system with div, curl etc. may be closer to the physics, or at least to our picture of it.

- In the vector notation system, we sometimes need to keep some vector formulas in memory or to consult a math handbook, while with the component system you need only the definitions of ε_{ijk} and δ_{ij}.

- Although not seen here, the component system of notation is more explicit (read unambiguous) when dealing with tensors of higher rank, for which vector notation becomes cumbersome.

- The vector notation system is independent of coordinate system, *i.e.*, $\nabla\phi$ is $\nabla\phi$ in any coordinate system, while in the component notation, the components depend on the unit vectors chosen.

Interpreting the continuity equation The equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0 \tag{1.51}$$

is known as a continuity equation. Why? Well, integrate the continuity equation over some volume V bounded by the surface S. By using Gauss's theorem, we find that

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_V \frac{\partial}{\partial t}\rho\,\mathrm{d}^3x = -\int_V \nabla \cdot \mathbf{j}\,\mathrm{d}^3x = -\int_S \mathbf{j} \cdot \mathrm{d}\mathbf{S} \tag{1.52}$$

which says that the change in the total charge in the volume is due to the net inflow of electric current through the boundary surface S. Hence, the continuity equation is the field theory formulation of the physical law of charge conservation.

END OF EXAMPLE 1.3◁

Electromagnetic Potentials and Waves

2.1 Coverage

Here we study the vector and scalar potentials $\mathbf{A}$ and ϕ and the concept of gauge transformation.

One of the most important physical manifestation of Maxwell's equations is the EM wave. Seen as wave equations, the Maxwell equations can be reduced to algebraic equations via the Fourier transform and the physics is contained in so-called dispersion relations which set the kinematic restrictions on the fields.

2.2 Formulae used

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

2.3 Solved examples

▷THE AHARONOV-BOHM EFFECT —————————————————————— EXAMPLE 2.1

Consider the magnetic field given in cylindrical coordinates,

$$\mathbf{B}(r < r_0, \theta, z) = B\hat{z} \tag{2.1}$$
$$\mathbf{B}(r > r_0, \theta, z) = 0 \tag{2.2}$$

Determine the vector potential **A** that "generated" this magnetic field.

Solution

A interesting question in electrodynamics is whether the EM potentials ϕ and **A** are more than mathematical tools, and alternatives to the Maxwell equations, from which we can derive the EM fields. Could it be that the potentials and not Maxwell's equations are more fundamental? Although the ultimate answer to these questions is somewhat metaphysical, it is exactly these questions that make the Aharonov-Bohm effect. Before we discuss this effect let us calculate the vector field from the given magnetic field.

The equations connecting the potentials with the fields are

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{2.3}$$
$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.4}$$

In this problem we see that we have *no boundary conditions* for the potentials. Also, let us use the gauge $\phi = 0$.

This problem naturally divides into two parts: the part within the magnetic field and the part outside the magnetic field. Let us start with the interior part:

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{0} \tag{2.5a}$$
$$\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} = 0 \tag{2.5b}$$
$$\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = 0 \tag{2.5c}$$
$$\frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right) = B \tag{2.5d}$$

The first equation tells us that **A** is time independent so $\mathbf{A} = \mathbf{A}(r,\theta,z)$. Examining the other three we find that there is no dependence on θ or z either so $\mathbf{A} = \mathbf{A}(r)$. All that remains is

$$\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} = B \tag{2.6}$$

Integrating this equation we find that

$$A_\theta = \frac{Br}{2} \tag{2.7}$$

Moving to the outer problem, we see that the only difference compared with the inner problem is that $\mathbf{B} = 0$ so that we must consider

$$\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} = 0 \tag{2.8}$$

This time integration leads to

$$A_\theta = \frac{C}{r} \tag{2.9}$$

If we demand continuity for the function A_θ over all space we find by comparing with (2.7) the arbitrary constant C and can write in outer solution as

$$A_\theta = \frac{B r_0^2}{2r} \neq 0\,! \tag{2.10}$$

Now in electrodynamics (read: in this course) the only measurable quantities are the fields. So the situation above, where we have a region in which the *magnetic field is zero* but the *potential is non-zero* has no measurable consequence in classical electrodynamics. In quantum mechanics however, the Aharonov-Bohm effect shows that this situation *does have a measurable consequence*. Namely, when letting charged particles go around this magnetic field (the particles are do not enter the magnetic field region because of a impenetrable wall) the energy spectrum of the particles after passing the cylinder will have changed, even though there is no magnetic field along their path. The interpretation is that the potential is a more fundamental quantity than the field.

──END OF EXAMPLE 2.1◁

▷INVENT YOUR OWN GAUGE────────────────────────────────────── EXAMPLE 2.2

Name some common gauge conditions and discuss the usefulness of each one. Then invent your own gauge and verify that it is indeed a gauge condition.

Solution

Background   The Maxwell equations that do not contain source terms can be "solved" by using the vector potential **A** and the scalar potential ϕ, defined through the relations

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{2.11}$$

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial t}\mathbf{A} \tag{2.12}$$

Assuming linear, isotropic and homogeneous media, we can use the constitutive relations $\mathbf{D} = \varepsilon\mathbf{E}\ \mathbf{H} = \mathbf{B}/\mu$, and $\mathbf{j} = \sigma\mathbf{E} + \mathbf{j}'$ (where $\mathbf{j}'$ is the free current density arising from other sources than conductivity) and definitions of the scalar and vector potentials in the remaining two Maxwell equations and find that

$$\nabla^2\phi + \boldsymbol{\nabla}\cdot\frac{\partial\mathbf{A}}{\partial t} = -\frac{\rho}{\varepsilon} \tag{2.13}$$

$$\nabla^2\mathbf{A} - \mu\sigma\frac{\partial}{\partial t}\mathbf{A} - \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \mu\varepsilon\frac{\partial}{\partial t}\phi + \mu\sigma\phi\right) = -\mu\mathbf{j}' \tag{2.14}$$

These equations are used to determine **A** and ϕ from the source terms. And once we have found **A** and ϕ it is straight forward to derive the **E** and **B** fields from (2.11) and (2.12).

The definitions of the scalar and vector potentials are not enough to make **A** and ϕ unique, i.e. , if one is given **A** and ϕ then (2.11) and (2.12) determine **B** and **E**, but if one is given **B** and **E** there many ways of choosing **A** and ϕ. This can be seen through the fact that **A** and ϕ can be transformed according to

$$\mathbf{A}' = \mathbf{A} + \nabla \psi \tag{2.15}$$

$$\phi' = \phi - \frac{\partial}{\partial t} \psi \tag{2.16}$$

where ψ is an arbitrary scalar field, but the **B** and **E** fields do not change. This kind of transformation is called a *gauge transformation* and the fact that gauge transformations do not affect the physically observable fields is known as *gauge invariance*.

Gauge conditions   The ambiguity in the definitions of **A** and ϕ can be used to introduce a *gauge condition*. In other words, since the definitions (2.11) and (2.12) do not completely define **A** and ϕ we are free to add certain conditions. Some common gauge conditions are

Coulomb gauge	$\nabla \cdot \mathbf{A} = 0$
Lorentz gauge	$\nabla \cdot \mathbf{A} + \mu \varepsilon \partial \phi / \partial t + \mu \sigma \phi = 0$
Temporal gauge	$\phi = 0$

The *Coulomb gauge* is most useful when dealing with static fields. Using $\nabla \cdot \mathbf{A} = 0$ then (2.13) and (2.14, for static fields, reduces to

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon} \tag{2.17}$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{j} \tag{2.18}$$

The *Lorentz gauge* is the most commonly used gauge for time-varying fields. In this case (2.13) and (2.14) reduce to

$$\left(\nabla^2 - \mu \sigma \frac{\partial}{\partial t} - \mu \varepsilon \frac{\partial^2}{\partial t^2} \right) \phi = -\frac{\rho}{\varepsilon} \tag{2.19}$$

$$\left(\nabla^2 - \mu \sigma \frac{\partial}{\partial t} - \mu \varepsilon \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu \mathbf{j} \tag{2.20}$$

So the Lorentz transform decouples (2.13) and (2.14) and puts ϕ and **A** on equal footing. Furthermore, the resulting equations are manifestly covariant.

In the *temporal gauge* one "discards" the scalar potential by setting $\phi = 0$. In this gauge (2.13) and (2.14) reduce to

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla \times \nabla \times \mathbf{A} = \mu \mathbf{j} \tag{2.21}$$

Thus the single vector **A** describes both **E** and **B** in the temporal gauge.

How to invent your own gauge   Gauges other than Coulomb, Lorentz and the temporal mentioned above are rarely used in introductory literature in Electrodynamics, but it is instructive to consider what constitutes a gauge condition by constructing ones own gauge.

Of course, a gauge condition is at least a scalar equation containing at least one of the components of **A** or ϕ. Once you have an equation that you think might be a gauge, it must be verified that it is a gauge. To verify that a condition is a gauge condition it is sufficient to show that any given set of **A** and ϕ can be made to satisfy your condition. This is done through gauge transformations. So given a **A** and a ϕ which satisfy the physical conditions through (2.13) and (2.14) we try to see if it is possible (at least in principle) to find a gauge transformation to some new potential **A**$'$ and ϕ', which satisfy your condition.

———————————————————————————END OF EXAMPLE 2.2◁

▷FOURIER TRANSFORM OF MAXWELL'S EQUATIONS—————————— EXAMPLE 2.3

Fourier transform Maxwell's Equation. Use the Fourier version of Maxwell's equations to investigate the possibility of waves that do not propagate energy; such waves are called *static waves*.

Solution

Maxwell's equations contain only linear operators in time and space. This makes it easy to Fourier transform them. By transforming them we get simple algebraic equations instead of differential equations. Furthermore, the Fourier transformed Maxwell equations are useful when working with waves or time-varying fields, especially since the response function, i.e. the dielectric function, is in many case more fundamentally described as a function of angular frequency ω than length **x**.

To perform this derivation we need formulas on how to translate the operators $\nabla\cdot$, $\nabla\times$ and $\partial/\partial t$ in Maxwell's equations.

The Fourier transform in time, is defined by

$$\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} dt \, e^{i\omega t} f(t) \tag{2.22}$$

and the Fourier transform in space, is defined analogously

$$\tilde{A}(\mathbf{k}) \equiv \int_{-\infty}^{\infty} d^3x \, e^{-i\mathbf{k}\cdot\mathbf{x}} A(\mathbf{x}) \tag{2.23}$$

and so a combined spatial and time Fourier transform becomes

$$\tilde{F}(\omega, \mathbf{k}) = \int_{-\infty}^{\infty} dt \, d^3x \, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} F(t, \mathbf{x}) \tag{2.24}$$

If we apply the last formula on $\nabla \cdot \mathbf{E}$ we get

$$
\begin{aligned}
\nabla \cdot \mathbf{E} \rightarrow & \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{d}^3x \, e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \nabla \cdot \mathbf{E} \\
= & \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{d}^3x \, e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \frac{\partial E_i(t,\mathbf{x})}{\partial x_i} \\
= & \int \mathrm{d}t \, \underbrace{\left[\sum_i E_i(t,\mathbf{x}) e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]_{\mathbf{x}\to\infty}}_{=0} - \int \mathrm{d}t \, \mathrm{d}^3x \, (-\mathrm{i}k_i E_i(t,\mathbf{x})) \\
= & \, \mathrm{i}k_i \int \mathrm{d}t \, \mathrm{d}^3x \, E_i(t,\mathbf{x}) \\
= & \, \mathrm{i}k_i \tilde{E}_i(\omega,\mathbf{k}) = \mathrm{i}\mathbf{k} \cdot \tilde{\mathbf{E}}(\omega,\mathbf{k})
\end{aligned}
\tag{2.25}
$$

where we have used partial integration. For $\nabla \times \mathbf{E}$ we get

$$
\begin{aligned}
\nabla \times \mathbf{E} \rightarrow & \int \mathrm{d}t \, \mathrm{d}^3x \, e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \nabla \times \mathbf{E}(t,\omega) \\
= & \int \mathrm{d}t \, \mathrm{d}^3x \, e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \varepsilon_{ijk} \frac{\partial E_j(t,\mathbf{x})}{\partial x_k} \mathbf{e}_i \\
= & \int \mathrm{d}t \, \underbrace{\left[E_i \ldots e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]}_{=0} - \int \mathrm{d}t \, \mathrm{d}^3x \, \left(-\mathrm{i}\varepsilon_{ijk}k_j E_k(t,\mathbf{x}) e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \mathbf{e}_i\right) \\
= & \, \mathrm{i}\varepsilon_{ijk} k_j \tilde{E}_k(\omega,\mathbf{k}) \mathbf{e}_i
\end{aligned}
\tag{2.26}
$$

where we have once again used partial integration. One may proceed analogously for $\partial/\partial t \mathbf{E}(t,\mathbf{x})$. Trivially, one gets similar equations for the transformation of the $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{B}$ fields. Thus we have found that

$$
\nabla \cdot \mathbf{V}(t,\mathbf{x}) \rightarrow \mathrm{i}\mathbf{k} \cdot \tilde{\mathbf{V}}(\omega,\mathbf{k}) \tag{2.27a}
$$

$$
\nabla \times \mathbf{V}(t,\mathbf{x}) \rightarrow \mathrm{i}\mathbf{k} \times \tilde{\mathbf{V}}(\omega,\mathbf{k}) \tag{2.27b}
$$

$$
\frac{\partial \mathbf{V}(t,\mathbf{x})}{\partial t} \rightarrow -\mathrm{i}\omega \tilde{\mathbf{V}}(\omega,\mathbf{k}) \tag{2.27c}
$$

where $\mathbf{V}(t,\mathbf{x})$ is an arbitrary field and $\rightarrow$ denotes here "Fourier transform". These transformation rules are easy to remember by saying that roughly the Fourier transform of ∇ is $\mathrm{i}\mathbf{k}$ and the Fourier transform of $\partial/\partial t$ is $-\mathrm{i}\omega$.

Now we can use (2.27a), (2.27b) and (2.27c) on Maxwell's equations. We then get, after some simple trimming

$$
\mathbf{k} \times \tilde{\mathbf{E}}(\omega,\mathbf{k}) = \omega \tilde{\mathbf{B}}(\omega,\mathbf{k}) \tag{2.28a}
$$

$$
\mathrm{i}\mathbf{k} \times \tilde{\mathbf{H}}(\omega,\mathbf{k}) = \tilde{\mathbf{j}}(\omega,\mathbf{k}) - \mathrm{i}\omega \tilde{\mathbf{D}}(\omega,\mathbf{k}) \tag{2.28b}
$$

$$
\mathbf{k} \cdot \tilde{\mathbf{D}}(\omega,\mathbf{k}) = -\mathrm{i}\tilde{\rho}(\omega,\mathbf{k}) \tag{2.28c}
$$

$$
\mathbf{k} \cdot \tilde{\mathbf{B}}(\omega,\mathbf{k}) = 0 \tag{2.28d}
$$

where we have dropped the ˜ notation. These are the Fourier versions of Maxwell's equation.

As an example of the use of the Fourier transformed Maxwell's equations let us derive static waves. Static waves are one possible oscillation mode for the **E** and **H** fields. Let's say that we have a mode α such that the $\mathbf{E} = \mathbf{E}_\alpha$ field is oscillating at $\omega = \omega_\alpha \neq 0$ and that it has a $\mathbf{k} = \mathbf{k}_\alpha \neq 0$ which is parallel to the electric field, so $\mathbf{k}_\alpha \parallel \mathbf{E}_\alpha$. From (2.28a) this implies that

$$\omega_\alpha \mathbf{B}_\alpha = \mathbf{k}_\alpha \times \mathbf{E}_\alpha = 0 \tag{2.29}$$

$$\mathbf{B}_\alpha = 0 \tag{2.30}$$

So, we see that $\mathbf{S} = \mathbf{E} \times \mathbf{H} = 0$ trivially. The lesson here is that you can have time-varying fields that do not transmit energy! These waves are also called *longitudinal* waves for obvious reasons.

————————————————————————————————————END OF EXAMPLE 2.3◁

▷SIMPLE DISPERSION RELATION————————————————————————— EXAMPLE 2.4

If a progressive wave is travelling in a linear, isotropic, homogenous, nonconducting dielectric medium with dielectricity parameter ε and permeability μ, what is the dispersion relation? And what is the group velocity in this case? Also, what is the dispersion relation in a conducting medium?

Solution

A dispersion relation is a relation between ω and **k**, usually something like

$$D(\omega, \mathbf{k}) = 0 \tag{2.31}$$

From this one can solve for ω which is then a function of (k_1, k_2, k_3). For isotropic media then ω will be a function of $|\mathbf{k}| = k$ only. A dispersion relation determines what modes (i.e. what combinations of **k** and ω) are possible. The dispersion relation is derivable in principal once one has explicit knowledge of the dielectricity function (or response function) for the medium in question.

The two vector equations in Maxwell's equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.32}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \tag{2.33}$$

so for a progressive wave characterized by ω and **k** propagating in a linear, isotropic, homogeneous medium with $\sigma = 0$ (so $\mathbf{j} = \sigma \mathbf{E} = 0$), then these equations give

$$\mathbf{k} \times \mathbf{E} = \omega\mathbf{B} \tag{2.34}$$

$$\mathbf{k} \times \frac{\mathbf{B}}{\mu} = -\omega\varepsilon\mathbf{E} \tag{2.35}$$

Operating on (2.34) with $\mathbf{k}\times$ and then using (2.35) we get a single vector equation:

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \omega\mathbf{k} \times \mathbf{B} \tag{2.36}$$

$$\Rightarrow \quad \underbrace{(\mathbf{k} \cdot \mathbf{E})}_{=0 \text{ Progressive wave!}} \mathbf{k} - k^2\mathbf{E} = -\omega^2\varepsilon\mu\mathbf{E} \tag{2.37}$$

$$\Rightarrow (k^2 - \omega^2\varepsilon\mu)\mathbf{E} = 0 \tag{2.38}$$

Since $\mathbf{E}$ is not assumed to be zero then

$$k^2 - \omega^2\varepsilon\mu = 0 \tag{2.39}$$

$$\Rightarrow \omega^2 = \frac{k^2}{\varepsilon\mu} \tag{2.40}$$

$$\Rightarrow \omega = \pm\frac{k}{\sqrt{\varepsilon\mu}} = \pm ku \tag{2.41}$$

where we have identified the phase velocity $u = 1/\sqrt{\varepsilon\mu}$.

The group velocity in this case is

$$v_g \equiv \frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}(uk) = u \tag{2.42}$$

so in this simple case the group velocity is the same as the phase velocity.

For the case of a conducting medium, in which $\mathbf{j} = \sigma\mathbf{E}$, the two vector equations applied on a wave which at first resembles the progressive wave we used above gives

$$\mathbf{k} \times \mathbf{E} = \omega\mathbf{B} \tag{2.43}$$

$$\mathbf{k} \times \frac{\mathbf{B}}{\mu} = -\mathrm{i}\sigma\mathbf{E} - \omega\varepsilon\mathbf{E} \tag{2.44}$$

Combining these two equation as done previously, we get

$$-k^2\mathbf{E} = -\mathrm{i}\sigma\mu\omega\mathbf{E} - \omega^2\varepsilon\mu\mathbf{E} \tag{2.45}$$

$$\Rightarrow \left(\omega^2 + \mathrm{i}\frac{\sigma}{\varepsilon}\omega - u^2k^2\right)\mathbf{E} = 0 \tag{2.46}$$

So that

$$\omega = \frac{\mathrm{i}\sigma}{2\varepsilon} \pm \frac{1}{2}\sqrt{4u^2k^2 - \frac{\sigma^2}{\varepsilon^2}} \tag{2.47}$$

and the group velocity is

$$\frac{\partial\omega}{\partial k} = \pm\frac{2u^2k}{\sqrt{4u^2k^2 - \sigma^2/\varepsilon^2}} \tag{2.48}$$

If $\sigma = 0$ then we are back again to the previous problem as can be verified.

END OF EXAMPLE 2.4◁

Relativistic Electrodynamics

3.1 Coverage

We examine the covariant formulation of electrodynamics. We take up the concept of 4-tensors and give examples of these. Also, show how 4-tensors are manipulated. We discuss the group of Lorentz transformations in the context of electrodynamics.

3.2 Formulae used

A Lorentz boost in the 3-direction

$$L^{\mu'}_{\mu} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{3.1}$$

The field tensor (components are 0, 1, 2, 3)

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{pmatrix} \tag{3.2}$$

3.3   Solved examples

▷COVARIANCE OF MAXWELL'S EQUATIONS─────────────────────────

Discuss the covariance of Maxwell's equations by showing that the wave equation for electromagnetic fields is invariant with respect to Lorentz transformations but not Galilean transformations.

Solution

The d'Alembert operator

$$\Box^2 \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{3.3}$$

is a fundamental operator in electrodynamics. The dynamics of EM fields is completely described using the d'Alembertian. Galilean transformations, even though closest to our intuitive picture of the fabric of space-time, does not leave the d'Alembertian invariant. A Galilean transformation is simply

$$\begin{cases} x_1' = x_1 \\ x_2' = x_2 \\ x_3' = x_3 - vt \\ t' = t \end{cases} \tag{3.4}$$

where the origin of the primed system is moving relative the unprimed along the 3-direction with velocity v. Now we introduce this transformation by expanding each differential in the unprimed coordinate system in terms of the differential in the primed system by using the chain rule of derivation, i.e. we evaluate $\partial/\partial x_\mu = (\partial x_\nu'/\partial x_\mu)\partial/\partial x_\nu'$, so

$$\frac{\partial}{\partial x_1} = \frac{\partial t'}{\partial x_1}\frac{\partial}{\partial t'} + \frac{\partial x_1'}{\partial x_1}\frac{\partial}{\partial x_1'} + \frac{\partial x_2'}{\partial x_1}\frac{\partial}{\partial x_2'} + \frac{\partial x_3'}{\partial x_1}\frac{\partial}{\partial x_3'} \tag{3.5}$$

$$= \frac{\partial}{\partial x_1'} \tag{3.6}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial t'}{\partial x_2}\frac{\partial}{\partial t'} + \frac{\partial x_1'}{\partial x_1}\frac{\partial}{\partial x_1'} + \frac{\partial x_2'}{\partial x_2}\frac{\partial}{\partial x_2'} + \frac{\partial x_3'}{\partial x_2}\frac{\partial}{\partial x_3'} \tag{3.7}$$

$$= \frac{\partial}{\partial x_2'} \tag{3.8}$$

$$\frac{\partial}{\partial x_3} = \frac{\partial t'}{\partial x_3}\frac{\partial}{\partial t'} + \frac{\partial x_1'}{\partial x_3}\frac{\partial}{\partial x_1'} + \frac{\partial x_2'}{\partial x_3}\frac{\partial}{\partial x_2'} + \frac{\partial x_3'}{\partial x_3}\frac{\partial}{\partial x_3'} \tag{3.9}$$

$$= \frac{\partial}{\partial x_3'} \tag{3.10}$$

and so

$$\frac{\partial^2}{\partial x_1^2} = \frac{\partial^2}{\partial x_1'^2}; \quad \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial x_2'^2}; \quad \frac{\partial^2}{\partial x_3^2} = \frac{\partial^2}{\partial x_3'^2} \tag{3.11}$$

but

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x_1'}{\partial t}\frac{\partial}{\partial x_1'} + \frac{\partial x_2'}{\partial t}\frac{\partial}{\partial x_2'} + \frac{\partial x_3'}{\partial t}\frac{\partial}{\partial x_3'} = \tag{3.12}$$

$$= \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x_3'} \tag{3.13}$$

and so

$$\frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x_3'}\right)\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x_3'}\right) = \frac{\partial^2}{\partial t'^2} + v^2\frac{\partial^2}{\partial x_3'^2} - 2v\frac{\partial^2}{\partial t'\partial x_3'} \tag{3.14}$$

where we have used the fact that the operators $\partial/\partial t'$ and $\partial/\partial x_3'$ commute. Thus we have found that

$$\Box^2 \to \Box^{2\prime} = \nabla'^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t'^2} - \frac{v^2}{c^2}\frac{\partial^2\Psi}{\partial x_3'^2} + \frac{2v}{c^2}\frac{\partial^2\Psi}{\partial t'\partial x_3'} = 0 \tag{3.15}$$

Which obviously does not have the same form as the d'Alembertian in the unprimed system!

Let us do the same calculations for the case of a Lorentz transformation; more specifically we consider a *boost* along the 3 axis which is given by

$$x^{\mu'} = L_{\mu}^{\mu'} x^{\mu} \tag{3.16}$$

where

$$L_{\mu}^{\mu'} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{3.17}$$

(remember that μ runs over 0, 1, 2, 3). Since γ and β depend on v The 4-gradient, $\partial_{\mu} \equiv \partial/\partial x^{\mu}$ transforms as

$$\partial_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\partial_{\mu'} = \frac{\partial}{\partial x^{\mu}}\left(L_{\nu}^{\mu'}x^{\nu}\right)\partial_{\mu'} = \left(x^{\nu}\frac{\partial L_{\nu}^{\mu'}}{\partial x^{\mu}} + L_{\nu}^{\mu'}\right)\partial_{\mu'} = L_{\mu}^{\mu'}\partial_{\mu'} \tag{3.18}$$

so

$$\Box^2 = \partial_{\mu}\partial^{\mu} = g^{\mu\nu}L_{\mu}^{\mu'}L_{\nu}^{\nu'}\partial_{\mu'}\partial_{\nu'}$$
$$= (\gamma^2 - \gamma^2\beta^2)\partial_0^2 - \partial_1^2 - \partial_2^2 - (\gamma^2 - \gamma^2\beta^2)\partial_3^2 = g^{\mu\nu}\partial_{\mu'}\partial_{\nu'} = \partial_{\mu'}\partial^{\mu'} \tag{3.19}$$

In other words we have found that the d'Alembertian is invariant under Lorentz boosts.

<div style="text-align: right">END OF EXAMPLE 3.1◁</div>

EXAMPLE 3.2   ▷INVARIANT QUANTITIES CONSTRUCTED FROM THE FIELD TENSOR——————
Construct the *dual tensor* $\mathscr{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ of the field tensor $F_{\mu\nu}$. What quantities constructed solely with the field tensor and it's dual tensor, are invariant under Lorentz transformations? Having found these quantities you should be able to answer the questions:

- can a purely electric field in one inertial system be seen as a purely magnetic field in another?

- and, can a progressive wave be seen as a purely electric or a purely magnetic field in an inertial system?

Solution

The dual tensor of $F_{\alpha\beta}$ is given by

$$\mathscr{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}g_{\gamma\mu}g_{\delta\nu}F^{\mu\nu} \tag{3.20}$$

where

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{pmatrix} \tag{3.21}$$

and

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.22}$$

We determine first the field tensor with two covariant indices through the formula

$$F_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}F^{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -cB_3 & cB_2 \\ -E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{pmatrix} \tag{3.23}$$

so

$$\mathscr{F}^{\alpha\beta} = \begin{pmatrix} 0 & -cB_1 & -cB_2 & -cB_3 \\ cB_1 & 0 & E_1 & -E_2 \\ cB_2 & -E_3 & 0 & E_1 \\ cB_3 & E_2 & -E_1 & 0 \end{pmatrix} \tag{3.24}$$

We see that the dual tensor can be obtained from $F^{\mu\nu}$ by putting $\mathbf{E} \to c\mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}/c$.

From the formula for the dual tensor $\mathscr{F}^{\mu\nu}$ we see that it is a 4-tensor since $F^{\mu\nu}$ is a four tensor and ε is easily shown to be an invariant under orthogonal transforms for which the Lorentz transform is a special case. What can we create from $\mathscr{F}^{\mu\nu}$ and $F^{\mu\nu}$ which is invariant under Lorentz transformations? We consider the obviously invariant quantities

$\mathscr{F}^{\mu\nu}F_{\mu\nu}$ and $F^{\mu\nu}F_{\mu\nu}$.

$$\mathscr{F}^{\mu\nu}F_{\mu\nu} = -4c\mathbf{E} \cdot \mathbf{B} \tag{3.25}$$

$$F^{\mu\nu}F_{\mu\nu} = 2(c^2B^2 - E^2) \tag{3.26}$$

This means that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - c^2B^2$ are Lorentz invariant scalars.

Relation of EM fields in different inertial systems   Now that we know that $\mathbf{E}\cdot\mathbf{B}$ and $E^2 - c^2B^2$ are Lorentz invariant scalars, let see what they say about EM fields in different inertial systems. Let us say that $X \equiv \mathbf{E} \cdot \mathbf{B}$ and $Y \equiv E^2 - c^2B^2$. All inertial systems must have the same value for X and Y. A purely electric field in one inertial system means that $\mathbf{B} = 0$, so $X = 0$ and $Y \geq 0$. A purely magnetic field would mean that $\mathbf{E}' = 0$, so $X = 0$ but $Y \leq 0$. In other words it does not seem that a purely electric field can be a purely magnetic field in any inertial system.

For a progressive wave $\mathbf{E} \perp \mathbf{B}$ so $X = 0$ and in a purely electric or a purely magnetic field $X = 0$ also, but for a progressive wave $E = cB$ so $Y = 0$ and if the other system has $\mathbf{E}' = 0$ or $\mathbf{B}' = 0$ then $Y = 0$ force both the fields to be zero. So this is not possible.

————————————————————————————————————END OF EXAMPLE 3.2◁

▷COVARIANT FORMULATION OF COMMON ELECTRODYNAMICS FORMULAS——————   EXAMPLE 3.3

Put the following well know formulas into a manifestly covariant form

- The continuity equation

- Lorentz force

- The inhomogeneous Maxwell equations

- The homogenous Maxwell equations

Solution

The Methodology   To construct manifestly covariant formulas we have at our disposal the following "building blocks":

$$
\begin{aligned}
\text{an event} \qquad & x^\mu = (ct, \mathbf{x}), \\
\text{4-velocity} \qquad & u^\mu = (\gamma c, \gamma\mathbf{v}), \\
\text{4-momentum} \qquad & p^\mu = (E/c, \mathbf{p}), \\
\text{wave 4-vector} \qquad & k^\mu = (\omega/c, \mathbf{k}), \\
\text{4-current density} \qquad & J^\mu = (\rho c, \mathbf{j}), \\
\text{4-potential} \qquad & A^\mu = (\phi/c, \mathbf{A}), \\
\text{4-force} \qquad & F^\mu = (\gamma\mathbf{v} \cdot \mathbf{F}/c, \gamma\mathbf{F})
\end{aligned}
$$

Also we have the 4-gradient

$$\text{4-gradient} \quad \partial_\mu \equiv \partial/\partial x^\mu = (\partial/\partial ct, \partial/\partial \mathbf{x})$$

as our operator building block and also the second rank 4-tensor

$$\text{field tensor} \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{pmatrix}$$

Observe that we use indices which run 0, 1, 2, 3 where the 0-component is time-like component, there is also the system where indices run 1, 2, 3, 4 and the 4-component is the time-like component. Beware!

A sufficient condition to formulate covariant electrodynamic formulas is that we make our formulas by combine the above 4-vectors. To make sure we have a covariant form we take outer product (i.e. simply combine the tensors so that all the indices are free) and then perform zero or more contractions, i.e. equate two indices and sum over this index (notationally this means we create a repeated index). In the notation we use here contractions must be between a contravariant (upper) index and a covariant (lower) index. One can always raise or lower a index by including a metric tensor $g_{\alpha\beta}$. On top of this sufficient condition, we will need to use our knowledge of the formulas we will try to make covariant, to accomplish our goal.

The continuity equation   We know that the continuity equation is a differential equation which includes the charge density and the current density and that it is a scalar equation. This leads us to calculate the contraction of the outer product between the 4-gradient ∂_μ and the 4-current J^ν

$$\partial_\mu J^\mu = 0 \tag{3.27}$$

This is covariant version of the continuity equation, thus in space-time the continuity equation is simply stated as the 4-current density is divergence-free!

Lorentz force   We know that the left hand side of Lorentz force equation is a 3-force. Obviously we should use the covariant 4-vector force instead. And on the right hand side of th Lorentz equation is a 3-vector quantity involving charge density and current density and the $\mathbf{E}$ and $\mathbf{B}$ fields. The EM fields are of course contained in the field tensor $F^{\mu\nu}$. To get a vector quantity from $F^{\mu\nu}$ and J^μ we contract these so our guess is

$$F^\mu = F^{\mu\nu} J_\nu \tag{3.28}$$

Inhomogeneous Maxwell equations   The inhomogeneous Maxwell may be written as the 4-divergence of the field tensor

$$\partial_\alpha F^{\alpha\beta} = J^\beta \tag{3.29}$$

Homogeneous Maxwell equations   The homogenous Maxwell equations are written most compactly using the dual tensor of the field tensor. Using the dual tensor we have

$$\partial_\alpha \mathscr{F}^{\alpha\beta} = 0 \tag{3.30}$$

▷FIELDS FROM UNIFORMLY MOVING CHARGE VIA LORENTZ TRANSFORMATION—— EXAMPLE 3.4

In the relativistic formulation of classical electrodynamics the **E** and **B** field vector form the antisymmetic electrodynamic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix} \tag{3.31}$$

Show that the fields from a charge q in uniform, rectilinear motion

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 s^3}(1 - v^2/c^2)\,\mathbf{R}_0$$

$$\mathbf{B} = \mathbf{v} \times \mathbf{E}/c^2$$

are obtained via a Lorentz transformation of the corresponding fields in the rest system of the charge.

Solution

We wish to transform the EM fields. The EM fields in a covariant formulation of electrodynamics is given by the electromagnetic field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -cB_3 & cB_2 & E_1 \\ cB_3 & 0 & -cB_1 & E_2 \\ -cB_2 & cB_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix} \tag{3.32}$$

where we are using components running as 1, 2, 3, 4. To transform the EM fields is to transform the field tensor. A Lorentz transformation of the field tensor can be written

$$F^{\lambda\sigma} = L^\lambda_\mu L^\sigma_\nu F^{\nu\mu} \tag{3.33}$$

$$\mathsf{F} = \mathsf{L}\mathsf{F}_0\mathsf{L} \tag{3.34}$$

where

$$L^\nu_\mu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{3.35}$$

where $\gamma = 1/\sqrt{1 - \beta^2}$ and $\beta = v/c$ where $\mathbf{v} = v_1\hat{\mathbf{x}}$. The fields in the rest system S^0 are

$$\mathbf{E}^0 = \frac{q}{4\pi\varepsilon_0} \frac{x^0\hat{\mathbf{x}} + y^0\hat{\mathbf{y}} + z^0\hat{\mathbf{z}}}{\sqrt{x^{02} + y^{02} + z^{02}}^3} \tag{3.36}$$

$$\mathbf{B}^0 = 0 \tag{3.37}$$

so that

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & E_1^0 \\ 0 & 0 & 0 & E_2^0 \\ 0 & 0 & 0 & E_3^0 \\ -E_1^0 & -E_2^0 & -E_3^0 & 0 \end{pmatrix} \tag{3.38}$$

A little matrix algebra gives

$$LFL = L \begin{pmatrix} 0 & 0 & 0 & E_1^0 \\ 0 & 0 & 0 & E_2^0 \\ 0 & 0 & 0 & E_3^0 \\ -E_1^0 & -E_2^0 & -E_3^0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} = \tag{3.39}$$

$$= \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -\beta\gamma E_1^0 & 0 & 0 & \gamma E_1^0 \\ -\beta\gamma E_2^0 & 0 & 0 & \gamma E_2^0 \\ -\beta\gamma E_3^0 & 0 & 0 & \gamma E_3^0 \\ -\gamma E_1^0 & -E_2^0 & -E_3^0 & \beta\gamma E_1^0 \end{pmatrix} = \tag{3.40}$$

$$= \begin{pmatrix} \beta\gamma^2 E_1^0 + \beta\gamma^2 E_1^0 & \gamma\beta E_2^0 & \gamma\beta E_3^0 & \gamma^2(1 - \beta^2)E_1^0 \\ -\gamma\beta E_2^0 & 0 & 0 & \gamma E_2^0 \\ -\gamma\beta E_3^0 & 0 & 0 & \gamma E_3^0 \\ -\gamma^2(1 - \beta^2)E_1^0 & -\gamma E_2^0 & -\gamma E_3^0 & -\beta\gamma^2 E_1^0 + \beta\gamma^2 E_1^0 \end{pmatrix} \tag{3.41}$$

$$E_1 = E_1^0 \tag{3.42}$$
$$E_2 = \gamma E_2^0 \tag{3.43}$$
$$E_3 = \gamma E_3^0 \tag{3.44}$$

$$B_1 = 0 \tag{3.45}$$
$$B_2 = \frac{\gamma\beta}{c}E_3^0 \tag{3.46}$$
$$B_3 = -\frac{\gamma\beta}{c}E_2^0 \tag{3.47}$$

$$\begin{pmatrix} x^0 \\ y^0 \\ z^0 \\ ct^0 \end{pmatrix} = L \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma(x - s) \\ y \\ z \\ \gamma(ct + \beta x) \end{pmatrix} \tag{3.48}$$

where $s = vt$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\gamma((x-s)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{\sqrt{\gamma^2(x-s)^2 + y^2 + z^2}^3} \tag{3.49}$$

$$R_0^2 = (s-x)^2 + y^2 + z^2 \tag{3.50}$$

$$s - x = R_0 \sin(\theta - \frac{\pi}{2}) = R_0 \cos\theta \tag{3.51}$$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0\gamma^2} \frac{\mathbf{R_0}}{\sqrt{(x-s)^2 + \frac{y^2+z^2}{\gamma^2}}^3} \tag{3.52}$$

$$(x-s)^2 + \frac{y^2 + z^2}{\gamma^2} = R_0^2 - y^2 - z^2 + \frac{y^2 + z^2}{\gamma^2} = \tag{3.53}$$

$$= R_0^2 + (y^2 + z^2)(\frac{1}{\gamma^2} - 1) = \tag{3.54}$$

$$= R_0^2(1 - \beta^2 \sin^2\theta) \tag{3.55}$$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}(1 - \beta^2)\frac{\mathbf{R_0}}{R_0^3\sqrt{1 - \beta^2 \sin^2\theta}^3} \tag{3.56}$$

END OF EXAMPLE 3.4◁

Lagrangian and Hamiltonian Electrodynamics

4.1 Coverage

We briefly touch the Lagrangian formulation of electrodynamics. We look at both the point Lagrangian for charges in EM fields and the Lagrangian density of the EM fields.

4.2 Formulae used

The Lagrangian for a charged particle in EM fields is

$$L = -\frac{mc^2}{\gamma} - q\phi + q\mathbf{v} \cdot \mathbf{A} \tag{4.1}$$

A useful Lagrangian density for EM field and its interaction with charged particles is given by

$$\mathscr{L} = -\frac{1}{2}\varepsilon_0(c^2 B^2 - E^2) - \mathbf{A} \cdot \mathbf{j} + \rho\phi \tag{4.2}$$

4.3   Solved examples

EXAMPLE 4.1   ▷CANONICAL QUANTITIES FOR A PARTICLE IN AN EM FIELD————————

Derive the canonical momentum and the generalised force for the case of a charged particle in EM field given by ϕ and $\mathbf{A}$. The Lagrangian is $L = -\frac{mc^2}{\gamma} - q\phi + q\mathbf{v} \cdot \mathbf{A}$.

Solution

We know from analytical mechanics that the canonical momentum $\mathbf{P}$ is found through

$$P_i = \frac{\partial L}{\partial v_i} \tag{4.3}$$

so with

$$L = -\frac{mc^2}{\gamma} - q\phi + q\mathbf{v} \cdot \mathbf{A} \tag{4.4}$$

we find that

$$
\begin{aligned}
P_i &= \frac{\partial L}{\partial v_i} \\
&= \frac{\partial}{\partial v_i} \left(-mc^2 \sqrt{1 - \frac{v_i v_i}{c^2}} - q\phi + q v_j A_j \right) \\
&= mc^2 \frac{v_i}{c^2 \sqrt{1 - \frac{v_i v_i}{c^2}}} + q A_i \\
&= m\gamma v_i + q A_i \tag{4.5} \\
\Rightarrow \quad &\mathbf{P} = \mathbf{p} + q\mathbf{A} \tag{4.6}
\end{aligned}
$$

And on the other hand the generalised force is

$$Q_i = -\frac{\partial U}{\partial x_i} + \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial U}{\partial \dot{x}_i} \tag{4.7}$$

where U is the generalised velocity dependent potential $q\phi - q\mathbf{v} \cdot \mathbf{A}$, which gives us,

$$
\begin{aligned}
Q_i &= -\frac{\partial}{\partial x_i} \left(q\phi - q v_j A_j \right) + \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial v_i} \left(q\phi - q v_j A_j \right) \\
&= -q\frac{\partial \phi}{\partial x_i} + q\frac{\partial v_j A_j}{\partial x_i} - q\frac{\mathrm{d}A_i}{\mathrm{d}t} \tag{4.8} \\
\Rightarrow \mathbf{Q} &= -q\boldsymbol{\nabla}\phi + q\boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) - q\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} \\
&= -q\boldsymbol{\nabla}\phi + q(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} + q(\mathbf{v} \times \boldsymbol{\nabla} \times \mathbf{A}) - q\frac{\partial \mathbf{A}}{\partial t} - q(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} \\
&= -q\boldsymbol{\nabla}\phi + q\mathbf{v} \times \boldsymbol{\nabla} \times \mathbf{A} + q\frac{\partial \mathbf{A}}{\partial t} \\
&= q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{4.9}
\end{aligned}
$$

What does the canonical momentum $\mathbf{P} = \mathbf{p} + q\mathbf{A}$ represent physically? Consider a charge moving in a static magnetic field. This charge will perform gyro-harmonic motion. Obviously the momentum is not conserved but on the other hand we did not expect it to be conserved since there is a force on the charge. However we now from analytical mechanics, that it is the conservation of *canonical* momentum that is more general. Conservation of canonical momentum is found when the problem is translational invariant, which is true in the case we have here since the potentials do not depend on spatial coordinates. So we expect $\mathbf{P} = \mathbf{p} + q\mathbf{A}$ to be a constant of the motion, but what is it?

───END OF EXAMPLE 4.1◁

▷GAUGE INVARIANCE OF THE LAGRANGIAN DENSITY───────────────────────── EXAMPLE 4.2

Consider the Lagrangian density of the EM fields in the form

$$\mathscr{L} = -\frac{1}{2}\varepsilon_0(c^2B^2 - E^2) - \mathbf{A} \cdot \mathbf{j} + \rho\phi \tag{4.10}$$

We know that EM field are invariant under gauge transformations

$$\mathbf{A} = \mathbf{A}' + \boldsymbol{\nabla}\psi \tag{4.11}$$

$$\phi = \phi' - \frac{\partial\psi}{\partial t} \tag{4.12}$$

Determine the Lagrangian density under a gauge transformation. Is it invariant? If not, discuss the consequences this would have.

Solution

Let us insert the gauge transformation relations into the Lagrangian density. Remembering that $\mathbf{E}$ and $\mathbf{B}$ are invariant under gauge transformations we find that

$$\mathscr{L} \to \mathscr{L}' = -\frac{1}{2}\varepsilon_0(c^2B^2 - E^2) - (\mathbf{A}' + \boldsymbol{\nabla}\psi) \cdot \mathbf{j} + \rho(\phi' - \frac{\partial\psi}{\partial t}) \tag{4.13}$$

$$-\frac{1}{2}\varepsilon_0(c^2B^2 - E^2) - \mathbf{A}' \cdot \mathbf{j} + \rho\phi' - (\boldsymbol{\nabla}\psi) \cdot \mathbf{j} - \rho\frac{\partial\psi}{\partial t} = \tag{4.14}$$

$$= -\frac{1}{2}\varepsilon_0(c^2B^2 - E^2) - \mathbf{A}' \cdot \mathbf{j} + \rho\phi' - \boldsymbol{\nabla} \cdot (\psi\mathbf{j}) + \psi(\boldsymbol{\nabla} \cdot \mathbf{j}) - \frac{\partial\rho\psi}{\partial t} + \psi\frac{\partial\rho}{\partial t} \tag{4.15}$$

$$= -\frac{1}{2}\varepsilon_0(c^2B^2 - E^2) - \mathbf{A}' \cdot \mathbf{j} + \rho\phi' + \psi(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}) - \boldsymbol{\nabla} \cdot (\psi\mathbf{j}) - \frac{\partial\rho\psi}{\partial t} \tag{4.16}$$

$$= \mathscr{L}_0 - \boldsymbol{\nabla} \cdot (\psi\mathbf{j}) - \frac{\partial\rho\psi}{\partial t} \tag{4.17}$$

where we have used the continuity equation. We see that $\mathscr{L}_0$ obviously has the same form as $\mathscr{L}$, but what about the other two terms?

Some thought reveals that the neccessary condition for a Lagrangian to be physically acceptable is not the Lagrangian itself is invariant but rather that the variation of the action integral $S = \int \int \mathscr{L} \, \mathrm{d}^3 x \, \mathrm{d}t$ is invariant. So now we would like to check that the gauge transformations indeed do not affect any the variation of the action. Now it is possible to proceed in two different ways to do this: one is simply carry out the integration in the definition of the action integral and check that its variation is zero, or two, remembering that the variation of the action is equivalent to the Euler-Lagrange equations, one could plug in the Lagrangian density (4.17) into Euler-Lagrange equations to check the resulting equations differ from the Maxwell equations.

Let us use the first alternative. Since the action is linear in $\mathscr{L}$ it is sufficient to examine

$$
\begin{aligned}
S_{\text{ext}} &= \int\!\!\int \left(\boldsymbol{\nabla} \cdot (\psi \mathbf{j}) + \frac{\partial \rho \psi}{\partial t} \right) \mathrm{d}^3 x \, \mathrm{d}t \\
&= \int\!\!\int \psi \mathbf{j} \cdot \mathrm{d}\mathbf{S} \, \mathrm{d}t + \left[\int \rho \psi \, \mathrm{d}^3 x \right]_{t_0}^{t_1} - \int\!\!\int \psi \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} \right) \mathrm{d}^3 x \, \mathrm{d}t \\
&= \int\!\!\int \psi \mathbf{j} \cdot \mathrm{d}\mathbf{S} \, \mathrm{d}t + \left[\int \rho \psi \, \mathrm{d}^3 x \right]_{t_0}^{t_1}
\end{aligned}
\tag{4.18}
$$

where we have used the continuity equation. Furthermore, if we assume no flux source/sink at infinity then we can write

$$
S_{\text{ext}} = \left[\int \rho \psi \, \mathrm{d}^3 x \right]_{t_0}^{t_1}
\tag{4.19}
$$

Now when taking the variation, we realize that we must hold t_0 and t_1 the end point of the particle path fixed, and thus

$$
\delta S_{\text{ext}} = 0
\tag{4.20}
$$

As one would expect, gauge transformations of the potentials do not effect the physics of the problem!

―――――――――――――――――――――――――――――――END OF EXAMPLE 4.2◁

Electromagnetic Energy, Momentum and Stress

5.1 Coverage

Here we study the force, energy, momentum and stress in an electromagnetic field.

5.2 Formulae used

Poynting's vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{5.1}$$

Maxwell's stress tensor

$$T_{ij} = E_i D_j + H_i B_j - \frac{1}{2}\delta_{ij}\left(E_k D_k + H_k B_k\right) \tag{5.2}$$

Table 5.1. The following table gives the relevant quantities. The field vectors in this table are assumed to be *real*. If the given fields are complex, use the real part in the formulas.

Name	Symbol	Formula	SI unit
Energy density	U_v	$\frac{1}{2}\mathbf{E}\cdot\mathbf{D}+\frac{1}{2}\mathbf{H}\cdot\mathbf{B}$	J/m^3
Intensity	$\mathbf{S}$	$\mathbf{E}\times\mathbf{H}$	W/m^2
Momentum density	$\mathbf{P}_{\mathrm{EM}}$	$\varepsilon\mu\mathbf{E}\times\mathbf{H}$	kg/s m^2
Stress	$\mathbf{T}$	$E_iD_j+H_iB_j-\frac{1}{2}\delta_{ij}(E_kD_k+H_kB_k)$	Pa

5.3   Solved examples

EXAMPLE 5.1   ▷EM QUANTITIES POTPOURRI

Determine the instantaneous values of the energy density, momentum density, intensity and stress associated with the electromagnetic fields for the following cases:

a) $\begin{cases} \mathbf{E}=E_0\mathbf{e}_2 e^{\mathrm{i}(k_1x_1-\omega t)} \\ \mathbf{H}=\frac{k}{\mu\omega}\hat{k}\times\mathbf{E} \end{cases}$ for $k_1=\sqrt{\varepsilon\mu}\,\omega$, $k_2=k_3=0$.

b) same as in a) but for $k_1=\mathrm{i}\alpha$

c) $\begin{cases} B_r=\frac{\mu_0 m}{2\pi}\frac{\cos\theta}{r^3} \\ B_\theta=\frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^3} \end{cases}$

Also, identify these cases. Assume that $\mathbf{D}=\varepsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$.

Solution

Background

(a) Progressive wave   This case is an example of a *progressive* or *propagating wave*. Since $\mathbf{E}$ and $\mathbf{H}$ are complex we must first take their real parts:

$$\mathrm{Re}\,\mathbf{E}=E_0\cos(k_1x_1-\omega t)\mathbf{e}_2 \tag{5.3}$$

$$\mathrm{Re}\,\mathbf{H}=-\frac{k_1}{\mu\omega}E_0\cos(k_1x_1-\omega t)\mathbf{e}_3 \tag{5.4}$$

The energy density is

$$U_v = \frac{\varepsilon}{2}E_0^2 \cos^2(k_1 x_1 - \omega t) + \frac{k_1^2}{2\mu\omega^2}E_0^2 \cos^2(k_1 x_1 - \omega t) = \tag{5.5}$$

$$= \varepsilon E_0^2 \cos^2(\omega\sqrt{\varepsilon\mu}x_1 - \omega t) \tag{5.6}$$

The intensity or power density is

$$\mathbf{S} = (E_0 \cos(k_1 x_1 - \omega t)\mathbf{e}_2) \times (-\frac{k_1}{\mu\omega}E_0 \cos(k_1 x_1 - \omega t)\mathbf{e}_3) = \tag{5.7}$$

$$= -\sqrt{\frac{\varepsilon}{\mu}}E_0^2 \cos^2(\omega\sqrt{\varepsilon\mu}x_1 - \omega t)\mathbf{e}_2 \times \mathbf{e}_3 = \tag{5.8}$$

$$= -\sqrt{\frac{\varepsilon}{\mu}}E_0^2 \cos^2(\omega\sqrt{\varepsilon\mu}x_1 - \omega t)\mathbf{e}_1 \tag{5.9}$$

The momentum is

$$\mathbf{P}_{\text{EM}} = \varepsilon\mu\mathbf{S} = \varepsilon\sqrt{\varepsilon\mu}E_0^2 \cos^2(\omega\sqrt{\varepsilon\mu}x_1 - \omega t) \tag{5.10}$$

The stress is

$$T_{11} = -\frac{\varepsilon}{2}E_2^2 - \frac{\mu}{2}H_3^2$$

$$= -\frac{\varepsilon}{2}E_0^2 \cos^2(k_1 x_1 - \omega t) - \frac{k_1^2}{\mu\omega^2}E_0^2 \cos^2(k_1 x_1 - \omega t)$$

$$= -\varepsilon E_0^2 \cos^2(k_1 x_1 - \omega t) \tag{5.11}$$

$$T_{21} = T_{31} = T_{12} = 0 \tag{5.12}$$

$$T_{22} = \frac{\varepsilon}{2}E_2^2 - \frac{\mu}{2}H_3^2$$

$$= \frac{\varepsilon}{2}E_0^2 \cos^2(k_1 x_1 - \omega t) - \frac{k_1^2}{\mu\omega^2}E_0^2 \cos^2(k_1 x_1 - \omega t) \tag{5.13}$$

$$T_{32} = T_{13} = T_{23} = 0 \tag{5.14}$$

$$T_{33} = \frac{\mu}{2}H_3^2 - \frac{\varepsilon}{2}E_2^2 = -T_{22} = 0 \tag{5.15}$$

(b) Evanescent wave   This case is a example of an *evanescent wave*. We take real part of the fields keeping in mind the fact that $k_1 = i\alpha$ is imaginary:

$$\text{Re}\{\mathbf{E}\} = E_0 e^{-\alpha x_1} \sin \omega t \, \mathbf{e}_2 \tag{5.16}$$

$$\text{Re}\{\mathbf{H}\} = -\frac{\alpha}{\mu\omega}\mathbf{e}_1 \times (\text{Im}\{\mathbf{E}\}) = -\frac{\alpha}{\mu\omega}\mathbf{e}_1 \times \left(E_0 e^{-\alpha x_1} \cos \omega t \, \mathbf{e}_2\right)$$

$$= -\frac{\alpha E_0}{\mu\omega}e^{-\alpha x_1} \cos \omega t \, \mathbf{e}_3 \tag{5.17}$$

The energy density is

$$U_v = \frac{\varepsilon}{2}E_0^2 e^{-2\alpha x_1}\sin^2 \omega t + \frac{\alpha^2 E_0^2}{2\mu\omega^2}e^{-2\alpha x_1}\cos^2 \omega t = \tag{5.18}$$

$$= \frac{E_0^2 e^{-2\alpha x_1}}{2}\left(\varepsilon\sin^2 \omega t + \frac{\alpha^2}{\mu\omega^2}\cos^2 \omega t\right) \tag{5.19}$$

The intensity is

$$\mathbf{S} = -\frac{\alpha E_0^2}{\mu\omega}e^{-2\alpha x_1}\sin \omega t \cos \omega t\mathbf{e}_1 \tag{5.20}$$

The momentum is

$$\mathbf{P}_{EM} = \varepsilon\mu\mathbf{S} = -\frac{\alpha E_0^2 \varepsilon}{\omega}e^{-2\alpha x_1}\sin \omega t \cos \omega t\mathbf{e}_1 \tag{5.21}$$

The stress is

$$T_{11} = -\frac{\varepsilon}{2}E_2^2 - \frac{\mu}{2}H_3^2 = \frac{E_0^2 e^{-2\alpha x_1}}{2}\left(\varepsilon\sin^2 \omega t + \frac{\alpha^2}{\mu\omega^2}\cos^2 \omega t\right) \tag{5.22}$$

$$T_{21} = T_{31} = T_{12} = 0 \tag{5.23}$$

$$T_{22} = \frac{\varepsilon}{2}E_2^2 - \frac{\mu}{2}H_3^2 = \frac{E_0^2 e^{-2\alpha x_1}}{2}\left(\varepsilon\sin^2 \omega t - \frac{\alpha^2}{\mu\omega^2}\cos^2 \omega t\right) \tag{5.24}$$

$$T_{32} = T_{13} = T_{23} = 0 \tag{5.25}$$

$$T_{33} = \frac{\mu}{2}H_3^2 - \frac{\varepsilon}{2}E_2^2 = -T_{22} = -\frac{E_0^2 e^{-2\alpha x_1}}{2}\left(\varepsilon\sin^2 \omega t - \frac{\alpha^2}{\mu\omega^2}\cos^2 \omega t\right) \tag{5.26}$$

(c) Magnetic dipole   This case is a magnetic dipole. The fields are real and in *spherical* coordinates.

The energy density is

$$U_v = \frac{1}{2\mu_0}(B_r^2 + B_\theta^2) = \frac{1}{2\mu_0}\left(\frac{\mu_0^2 m^2}{16\pi^2}\frac{4\cos^2 \theta}{r^6} + \frac{\mu_0^2 m^2}{16\pi^2}\frac{\sin^2 \theta}{r^6}\right)$$

$$= \frac{\mu_0 m^2}{32\pi^2}\left(4\cos^2 \theta + \sin^2 \theta\right)\frac{1}{r^6}$$

$$= \frac{\mu_0 m^2}{32\pi^2 r^6}(1 + 3\cos^2 \theta) \tag{5.27}$$

The intensity is $\mathbf{S} = 0$ since $\mathbf{E} = 0$, and likewise for the momentum. The stress tensor components are

$$T_{rr} = \frac{1}{\mu_0}B_r^2 - U_v = \frac{\mu_0 m^2}{4\pi^2}\frac{\cos^2\theta}{r^6} - \frac{\mu_0 m^2}{32\pi^2 r^6}(1 + 3\cos^2\theta)$$

$$= \frac{\mu_0 m^2}{32\pi^2 r^6}(8\cos^2\theta - 1 - 3\cos^2\theta)$$

$$= -\frac{\mu_0 m^2}{32\pi^2 r^6}(1 - 5\cos^2\theta) \tag{5.28}$$

$$T_{\theta r} = \frac{1}{\mu_0}B_r B_\theta = \frac{\mu_0 m^2}{8\pi}\frac{\sin\theta\cos\theta}{r^6} \tag{5.29}$$

$$T_{\phi r} = 0 \tag{5.30}$$

$$T_{r\theta} = T_{\theta r} \tag{5.31}$$

$$T_{\theta\theta} = \frac{1}{\mu_0}B_\theta^2 - U_v = \frac{\mu_0 m^2}{16\pi^2}\frac{\sin^2\theta}{r^6} - \frac{\mu_0 m^2}{32\pi^2 r^6}(1 + 3\cos^2\theta)$$

$$= \frac{\mu_0 m^2}{32\pi^2 r^6}(2\sin^2\theta - 1 - 3\cos^2\theta)$$

$$= -\frac{\mu_0 m^2}{32\pi^2 r^6}(1 - 3\cos^2\theta + 2\sin^2\theta) \tag{5.32}$$

$$T_{\phi\theta} = 0 \tag{5.33}$$

$$T_{r\phi} = 0 \tag{5.34}$$

$$T_{\theta\phi} = 0 \tag{5.35}$$

$$T_{\phi\phi} = 0 \tag{5.36}$$

———END OF EXAMPLE 5.1◁

▷CLASSICAL ELECTRON RADIUS———————————————————————————— | EXAMPLE 5.2

Calculate the classical radius of the electron by assuming that the mass of an electron is the mass of it's electric field and that the electron is a homogenous spherical charge distribution of radius r_e and total charge $|q| = e$. Mass and energy are related through the equation $E = mc^2$.

Solution

One the "failures" of Maxwell's equations or classical electrodynamics is on question of mass of point particles. From relativity one can show that fundamental particles should be point-like. However, if one calculates the electromagnetic mass from Maxwell's equation one gets an infinite result due to the singularity in Gauss law $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$. This points to the fact that Maxwell's equations has a minimum length scale validity where quantum mechanics takes over.

We will calculate this problem as follows: determine the electric field in all of space and then integrate the formula for the energy density of the electric field. This integral will contain the radius of the electron since it partitions the integration. We then relate this field energy to the mass of the electron, which is a known quantity. We use this relation to solve

for the "electron radius".

We start by determining the electric field. Consider a homogeneous spherical charge distribution in a vacuum. Introduce spherical coordinates. We divide the whole space into two parts: the region outside of the sphere, i.e. $r > r_e$, is given by Coulomb's law:

$$E(r > r_e) = \frac{e}{4\pi\varepsilon_0 r^2} \tag{5.37}$$

and the region inside the sphere is given by Gauss law

$$\nabla \cdot \mathbf{E}(r < r_e) = \rho/\varepsilon_0 \tag{5.38}$$

which in this case gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = \frac{e}{4/3\pi r_e^3\,\varepsilon_0} \tag{5.39}$$

If we integrate this equation, i.e. (5.39), we find that

$$E_r(r < r_e) = \frac{e}{4\pi r_e^3\varepsilon_0} r \tag{5.40}$$

We can easily verify that (5.40) is indeed a solution to (5.39) and furthermore it is continuous with (5.37), thus making the solution unique.

We may now determine the energy density of the electric field due to the electron. It is simply

$$U_v = \frac{1}{2}\varepsilon_0 E^2 \tag{5.41}$$

$$\Rightarrow \begin{cases} U_v(r > r_e) = \frac{e^2}{32\pi^2\varepsilon_0}\frac{1}{r^4} \\ U_v(r < r_e) = \frac{e^2}{32\pi^2\varepsilon_0 r_e^6} r^2 \end{cases} \tag{5.42}$$

Now we integrate U_v over all space. That is

$$\begin{aligned} U &= \int_{\text{all space}} U_v(r < r_e) + U_v(r > r_e)\, \mathrm{d}^3x \\ &= \int_0^{4\pi}\int_0^{r_e} \frac{e^2}{32\pi^2\varepsilon_0 r_e^6} r^2\, r^2 \mathrm{d}r\, \mathrm{d}\Omega + \int_0^{4\pi}\int_{r_e}^{\infty} \frac{e^2}{32\pi^2\varepsilon_0}\frac{1}{r^4} r^2\, \mathrm{d}r\, \mathrm{d}\Omega \\ &= \frac{e^2}{8\pi\varepsilon_0 r_e^6}\left[\frac{r^5}{5}\right]_0^{r_e} + \frac{e^2}{8\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_e}^{\infty} = \frac{3e^2}{20\pi\varepsilon_0}\frac{1}{r_e} \end{aligned} \tag{5.43}$$

Finally, we relate the total electric field energy to the rest mass of the electron and solve for the electron radius,

$$U = m_e c^2 \tag{5.44}$$

$$\Rightarrow \frac{3e^2}{20\pi\varepsilon_0}\frac{1}{r_e} = m_e c^2$$

$$\Rightarrow r_e = \frac{3}{5}\frac{e^2}{4\pi\varepsilon_0 m_e c^2} \tag{5.45}$$

This last result can be compared with the de facto *classical electron radius* which is defined as

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} \tag{5.46}$$

and is found by calculating the scattering cross section of the electron.

— END OF EXAMPLE 5.2 ◁

▷ SOLAR SAILING ————————————————————————————

EXAMPLE 5.3

Investigate the feasibility of sailing in our solar system using the solar wind. One technical proposal uses kapton, 2 mm in thickness, with a 0.1 mm thick aluminium coating as material for a sail. It would weigh 1 g/m². At 1 AU (= astronomical unit ~ distance between sun and earth) the intensity of the electromagnetic radiation from the sun is of the order 1.4×10^3 W/m². If we assume the sail to be a perfect reflector, what would the acceleration be for different incidence angles of the sun's EM radiation on the sail?

Solution

In this problem we will use the principle of momentum conservation. If the continuity equation for electromagnetic momentum $\partial \mathbf{P}^{\text{EM}}/\partial t + \nabla \cdot \mathbf{T} = \partial \mathbf{P}^{\text{mech}}/\partial t$ is integrated over all space the stress term disappears and what is left says that a change in electromagnetic momentum is balanced by mechanical force.

Imagine a localised pulse of a progressive electromagnetic wave incident on a plane. That the wave is progressive means simply that it is "purely radiation" (more about progressive waves in the next lesson). Let us characterize this wave by a Poynting vector $\mathbf{S}$ of duration Δt. It travels in space with velocity c, it "lights up" the area ΔA on the surface on the sail, and, furthermore, $\mathbf{S}$ makes an angle $180° - \theta$ with the normal of the sail surface. The momentum carried by such a wave is

$$\mathbf{p}_{\text{EM}}^{\text{before}} = \mathbf{P}_{\text{EM}}^{\text{before}} (\Delta V) = \mathbf{S}/c^2 \, (\Delta A \, c \, \Delta t \, \cos\theta) \tag{5.47}$$

so the momentum along the direction of the surface normal $\hat{\mathbf{n}}$ is

$$\mathbf{p}_{\text{EM}}^{\text{before}} \cdot \hat{\mathbf{n}} = -\frac{|\mathbf{S}|}{c} \Delta A \, \Delta t \, \cos^2\theta \tag{5.48}$$

After hitting the sail, the pulse will be characterized with all the same quantities as before the impact, except that the component along the surface normal will be opposite in sign. This is because the sail is assumed to be perfectly reflecting. Thus,

$$\mathbf{p}_{\text{EM}}^{\text{after}} \cdot \mathbf{n} = -\mathbf{p}_{\text{EM}}^{\text{before}} \cdot \mathbf{n} = \frac{|\mathbf{S}|}{c} \Delta A \, \Delta t \, \cos^2\theta \tag{5.49}$$

Now the continuity equation for EM momentum says that $\partial \mathbf{P}_{\text{EM}}/\partial t = \partial \mathbf{P}_{\text{mech}} = \mathbf{F}$ so for the component along the sail surface normal this implies that

$$\mathbf{F} \cdot \hat{\mathbf{n}} = \frac{\left(\mathbf{p}_{\text{EM}}^{\text{after}} - \mathbf{p}_{\text{EM}}^{\text{before}} \right)}{\Delta t} \cdot \hat{\mathbf{n}} = 2 \frac{|\mathbf{S}|}{c} \Delta A \, \cos^2 \theta \tag{5.50}$$

The other force components are zero by symmetry.

So finally, the pressure $\mathscr{P}$ exerted by the pulse on the sail is simply

$$\mathscr{P} = \frac{\mathbf{F} \cdot \hat{\mathbf{n}}}{\Delta A} = 2 \frac{|\mathbf{S}|}{c} \cos^2 \theta \tag{5.51}$$

We now have the pressure exerted by a pulse charcterized by $\mathbf{S}$ incident on a surface with an angle θ. The solar wind can be seen as a multitude of such pulses radiating radially outwards from the sun. At 1 AU, the solar constant is 1.3 kW/m^2. The solar constant is the intensity of EM radiation or in other words, the magnitude of the Poynting vector $|\mathbf{S}|$. From equation (5.51), we find that

$$\mathscr{P} = 2 \frac{1.3 \times 0^3 \, \text{W/m}^2}{3 \times 10^8 \, \text{m/s}} \cos^2 \theta \tag{5.52}$$

Newton's second law gives the acceleration a of the sail (which, as we recall, weighs 3×10^{-3} kg/m^2) due to the solar wind can at most be

$$a_\perp = \frac{0.9 \times 10^{-5} \, \text{N/m}^2}{3 \cdot 10^{-3} \, \text{kg/m}^2} = 3 \, \text{mm/s}^2 \tag{5.53}$$

which is half of the acceleration due to the sun's gravitational field.

END OF EXAMPLE 5.3◁

EXAMPLE 5.4

▷MAGNETIC PRESSURE ON THE EARTH

Determine the magnetic pressure due to the Earth's magnetic field at the magnetic poles (take $B = 6 \times 10^{-5}$ T) and compare this with the Earth's atmospheric pressure (1 atm $= 1.01 \times 10^5$ Pa). Now assume that the magnetic dipole moment is proportional to the angular velocity of the earth, how much faster would the angular velocity need to be for the magnetic pressure to be comparable to the atmospheric pressure?

Solution

In this exercise we see that EMF can exert pressure not only via radiation pressure, but also through static fields.

The static magnetic pressure is quantified in the Maxwellian stress tensor, which for this case is

$$
\mathbf{T} = \begin{bmatrix} B_r^2/\mu_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} U_v & 0 & 0 \\ 0 & U_v & 0 \\ 0 & 0 & U_v \end{bmatrix} . \tag{5.54}
$$

So by taking the inner product of $\mathbf{T}$ and the unit vector pointing in the direction the north pole, namely $[1\ 0\ 0]$, we find the pressure in the radial direction to be

$$
\mathscr{P} = \frac{B_r^2}{2\mu_0} = \frac{(6 \cdot 10^{-5})^2}{1.3 \cdot 10^{-6}} 2 \cdot 1.3 \cdot 10^{-6} = 1.4 \cdot 10^{-3}\,\text{Pa} \tag{5.55}
$$

Now we assume, according to the hypothesis in problem, that

$$
\frac{1}{T_{\text{period}}} \propto m \propto B \propto \sqrt{\mathscr{P}} \tag{5.56}
$$

so that

$$
\frac{\mathscr{T}_1}{\mathscr{T}_0} = \sqrt{\frac{P_0}{P_1}} \tag{5.57}
$$

where we denote the current values of the pressure and rotational period time, $\mathscr{P}_0$ and T_0 respectively and the hypothetical values $\mathscr{P}_1$ and T_1. Solving for T_1 we arrive at

$$
\Rightarrow T_1 = T_0 \sqrt{\frac{\mathscr{P}_0}{\mathscr{P}_1}} \tag{5.58}
$$

So with $\mathscr{P}_1 = 10^5$ Pa, we find that

$$
T_1 = 24 \cdot 3600 \cdot \sqrt{\frac{1.4 \cdot 10^{-3}}{10^5}} = 10\,\text{s} \tag{5.59}
$$

END OF EXAMPLE 5.4◁

Radiation from Extended Sources

6.1 Coverage

We will study the important general problem of how to calculate the EM fields induced by spatially extended, time-varying sources. This problem is solved in different ways depending on the explicit form of the source distribution. For truly extended bodies with non-monochromatic time dependence, we use the general expressions for the retarded potentials. And for monochromatic, one-dimensional current distributions, which we will call *antennas* here, we use the formulas given below.

6.2 Formulae used

General expressions for retarded potentials

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{j}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}$$

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}$$

For "antennas", we use

$$\mathbf{B}_\omega = \frac{-i\mu_0 k \sin\theta}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \mathscr{I} \hat{\boldsymbol{\varphi}}$$

$$\mathbf{E}_\omega = \frac{-ik \sin\theta}{4\pi\varepsilon_0 c} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \mathscr{I} \hat{\boldsymbol{\theta}}$$

$$\langle \mathbf{S} \rangle = \frac{k^2 \sin^2 \theta \mu_0 c}{32\pi^2 |\mathbf{x} - \mathbf{x}'|^2} |\mathscr{I}|^2 \hat{\mathbf{r}}$$

where

$$\mathscr{I} = \int_{-l}^{l} j_\omega(z') e^{-ikz' \cos \theta} dz'$$

6.3   Solved examples

EXAMPLE 6.1   ▷INSTANTANEOUS CURRENT IN AN INFINITELY LONG CONDUCTOR——————

Consider the following idealised situation with an infinitely long, thin, conducting wire along the z axis. For $t < 0$, it is current free, but at time $t = 0$ a constant current J is applied simultaneously over the entire length of the wire. Consequently, the wire carries the current

$$j(z) = \begin{cases} 0, & t < 0 \\ J, & t \ge 0 \end{cases}$$

It is assumed that the conductor can be kept uncharged, *i.e.*, $\rho = 0$. Determine $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{S}$ in whole space.

Hint: Calculate first the vector potential $\mathbf{A}$.

Solution

This problem belongs to the most general category of problems of the kind where given a source distribution one wants to find the EM fields. This is because the source is not monochromatic, so it is not an antenna, and furthermore it is an extended distribution, so multipole expansion analysis is not possible. So we must use the most general formula for calculating fields induced by time-varying sources, which in the Lorentz gauge take the form

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{j}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{6.1}$$

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{6.2}$$

where the *source time* t' is to be replaced by $t' = t - |\mathbf{x} - \mathbf{x}'|/c$ so t' is seen as a function of t, $\mathbf{x}$ and $\mathbf{x}'$. The function $t - |\mathbf{x} - \mathbf{x}'|/c$ is known as the retarded time t'_{ret}, an expression which is meaningful only relative the field point $(t, \mathbf{x})$.

The algorithm of solution now that we have decided to use (6.1) is to find the explicit form for $j(t', \mathbf{x}')$ and $\rho(t', \mathbf{x}')$, perform the integrations to obtain the potentials $\mathbf{A}(t, \mathbf{x})$ and $\phi(t, \mathbf{x})$ and, finally, derive the $\mathbf{E}$ and $\mathbf{B}$ fields from the potentials in the normal fashion and the Poynting vector from the fields.

Let us find the explicit expression for the current density $j(\mathbf{x}', t')$, which is illustrated in Fig. 6.1. In many problems, the expressions for the sources consist of a time-dependent part times a space-dependent part. This is one such case. The "switching on" at $t' = 0$ can be written as a step function $\theta(t')$. And if we orient the wire along the z axis, the space dependent part can be written $\delta(x')\delta(y')J\hat{z}$.

On the other hand, the charge distribution $\rho(t, \mathbf{x}) = 0$ as given in the problem formulation. This can be seen as $+$ and $-$ charges flowing in opposite directions such as to keep the total charge density $\rho(t, \mathbf{x}) = 0$ but this could still have a total current density $\mathbf{j}(t, \mathbf{x}) \neq 0$.

So we have for the current that

$$\mathbf{j}(t', \mathbf{x}') = \delta(x')\delta(y')J\theta(t')\hat{z} \tag{6.3}$$

and for the charge density

$$\rho(t', \mathbf{x}') = 0 \tag{6.4}$$

We insert (6.3) into (6.1), remembering to replace the source time t' with $t - |\mathbf{x} - \mathbf{x}'|/c$, and perform the integration. The integrations over x' and y' are trivial:

$$\mathbf{A}(t, \mathbf{x}) = \hat{z}\frac{\mu_0 J}{4\pi} \int_{-\infty}^{\infty} dz' \frac{\theta(t - |\mathbf{x} - z'\hat{z}|/c)}{|\mathbf{x} - z'\hat{z}|} \tag{6.5}$$

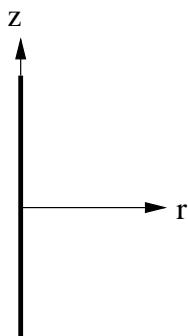Figure 6.1. The current density distribution $\mathbf{j}$ is along the z axis and is turned on at $t' = 0$. We use cylindrical coordinates.

For the remaining z' integration we use cylindrical coordinates (see Figure BLP1.2cyl) so we can write

$$\mathbf{x} = r\hat{\mathbf{r}} \tag{6.6}$$

$$\Rightarrow |\mathbf{x} - \mathbf{x}'| = \sqrt{r^2 + z'^2}. \tag{6.7}$$

The step function in the integrand is zero when its argument is less than zero. This means that when integrating over z', only those z' contribute which satisfy

$$t - |r\hat{\mathbf{r}} - z'\hat{\mathbf{z}}|/c > 0 \;\Rightarrow\; ct > \sqrt{r^2 + z'^2} \tag{6.8}$$

which can be written, if we assume

$$t > 0, \tag{6.9}$$

as

$$\Rightarrow z'^2 + r^2 < c^2 t^2 \tag{6.10}$$

$$\Rightarrow z' < |\sqrt{c^2 t^2 - r^2}| \tag{6.11}$$

or

$$-\sqrt{c^2 t^2 - r^2} < z' < \sqrt{c^2 t^2 - r^2} \equiv a \tag{6.12}$$

where we have introduced a simply as a shorthand. These limits can be understood as follows (*cf.* Fig. 6.2): after the current is switched on, EM fields are sent from each point along the wire and travel at the speed of light. Since the information (*i.e.*, the current turn-on) carried by the fields travel in the "line of sight", or in other words in a straight line, each field point only sees those parts of the current which are close enough. This illustrates the concept of retarded time, which is only meaningful relative the field point, and also the *information gathering sphere*.

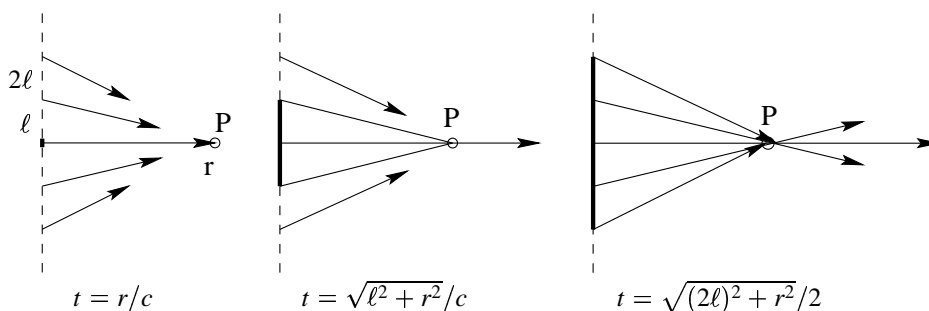Figure 6.2. This series of snapshots shows what the part of the current is seen at the field point P at different field times t.

So we have now that

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu_0 J}{4\pi} \int_{-a}^{a} \frac{1}{\sqrt{z'^2 + r^2}} dz'$$

$$= \hat{\mathbf{z}}\frac{\mu_0 J}{4\pi} \ln \left(\frac{1 + \sqrt{1 - \frac{r^2}{c^2 t^2}}}{1 - \sqrt{1 - \frac{r^2}{c^2 t^2}}} \right) \tag{6.13}$$

On the other hand, the scalar potential we may set so $\phi(t, \mathbf{x}) = 0$, since $\rho(t', \mathbf{x}') = 0$.

Now that we have the vector potential we can derive the $\mathbf{E}$ and $\mathbf{B}$ fields. The $\mathbf{B}$ field is

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.14}$$

and since $\mathbf{A}$ only has its $\hat{z}$-component different from zero in the cylindrical system, we have

$$\mathbf{B} = -\hat{\boldsymbol{\varphi}} \frac{\partial A_z}{\partial r} \tag{6.15}$$

Thus we only need $\frac{\partial A_z}{\partial r}$. Introducing $f \equiv \sqrt{1 - r^2/(c^2 t^2)}$, which satisfies

$$\frac{\partial f}{\partial r} = -\frac{r}{c^2 t^2} \frac{1}{f} \tag{6.16}$$

we then can write

$$\begin{aligned}
\frac{\partial A_z}{\partial r} &= \frac{\mu_0 J}{4\pi} \frac{\partial}{\partial r} \left(\ln \left(\frac{1+f}{1-f} \right) \right) \\
&= -\frac{\mu_0 J}{4\pi} \frac{2}{rf}
\end{aligned} \tag{6.17}$$

$$\Rightarrow \mathbf{B} = \hat{\boldsymbol{\varphi}} \frac{\mu_0 J}{2\pi r \sqrt{1 - \frac{r^2}{c^2 t^2}}} \tag{6.18}$$

Observe that for large t we have the "static" case: $\mathbf{B} = \hat{\boldsymbol{\varphi}} \mu_0 J/(2\pi r)$

The electric field is derived from

$$\begin{aligned}
\mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} \\
&= -\hat{z} \frac{\mu_0 J}{4\pi} \frac{\partial}{\partial t} \left(\ln \left(\frac{1+f}{1-f} \right) \right)
\end{aligned} \tag{6.19}$$

But

$$\frac{\partial f}{\partial t} = \frac{1}{2f} \frac{2r^2}{c^2 t^3} = \frac{1}{f} \frac{r^2}{c^2 t^3} \tag{6.20}$$

and so

$$\frac{\partial}{\partial t} \left(\ln \left(\frac{1+f}{1-f} \right) \right) = \frac{1-f}{1+f} \frac{2\frac{\partial f}{\partial t}}{(1-f)^2} = \frac{2}{ft} \tag{6.21}$$

and

$$\mathbf{E} = -\hat{z} \frac{\mu_0 J}{2\pi} \frac{1}{t\sqrt{1 - \frac{r^2}{c^2 t^2}}} \tag{6.22}$$

Notice that as $t \to \infty$ then $\mathbf{E} \to 0$!

All we have left is to determine the Poynting vector $\mathbf{S}$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}\frac{\mu_0 J}{2\pi r\sqrt{1-\frac{r^2}{c^2 t^2}}}\frac{\mu_0 J}{2\pi t\sqrt{1-\frac{r^2}{c^2 t^2}}}(\hat{z}\times\hat{\boldsymbol{\varphi}}) \tag{6.23}$$

$$= \frac{\mu_0}{(2\pi)^2}\frac{J^2}{rt(1-\frac{r^2}{c^2 t^2})}\hat{\boldsymbol{r}} \tag{6.24}$$

From this expression we can for example calculate the radiated power per unit length by integrating over a cylindrical surface $\mathbf{C}$ enclosing the wire:

$$P = \oint \mathbf{S}\cdot d\mathbf{C}$$

$$= \frac{\mu_0}{2\pi}\frac{J^2}{t(1-\frac{r^2}{c^2 t^2})} \tag{6.25}$$

We see that an infinite power is transmitted starting at $t = 0$ and $r = 0$ which travels out to infinite r. So in practice it is impossible to produce this physical setup. This is due to the quick "turn on". In the physical world only gradual turn ons are possible. This many be seen as a consequence of what is known as *Gibb's phenomenom*.

In the above we tacitly employed the retarded potential without discussing the possibility of using the advanced potential. Let us see what happens if we apply the advanced potential to this problem. The only thing that changes from the outset is that the source time t' is replaced by the advanced time $t_a \equiv t + |\mathbf{x} - \mathbf{x}'|/c$ instead of the retarded time $t_r \equiv t - |\mathbf{x} - \mathbf{x}'|/c$. With t_a as the argument to the step function, the contribution to the integral comes only from those z' which satisfy

$$t + |r\hat{\boldsymbol{r}} - z'\hat{\boldsymbol{z}}|/c > 0 \Rightarrow -t < |r\hat{\boldsymbol{r}} - z'\hat{\boldsymbol{z}}|/c \tag{6.26}$$

If we assume that

$$t < 0, \tag{6.27}$$

then we may write this as

$$\Rightarrow z'^2 + r^2 > c^2 t^2 \tag{6.28}$$

$$\Rightarrow z' > \pm\sqrt{c^2 t^2 - r^2} \tag{6.29}$$

or

$$z' < -\sqrt{c^2 t^2 - r^2},\ \sqrt{c^2 t^2 - r^2} < z' \tag{6.30}$$

One can proceed further and calculate the resulting integral. But what is interesting is that now we see that the relation (6.9) seems to say that we have no information about what happened *before* turn on, while the relation (6.27) says we have no information about what happened after turn on. Physics seems to be conspiring on us in such a way that we cannot compare the advanced and the retarded potential at the *same* time!

<div style="text-align:right">END OF EXAMPLE 6.1◁</div>

EXAMPLE 6.2

▷MULTIPLE HALF-WAVE ANTENNA——————————————————————————

A wire antenna with a length of $m(\in \mathbb{N})$ half wavelengths is assumed to have a current distribution in the form of a standing wave with current nodes at the its endpoints. Determine the angular distribution of the radiated electromagnetic power from the antenna.

Hint: It can me convenient to treat even m and odd m separately.

Solution

One realises that the setup in this problem is an antenna, since we have a monochromatic source and the current is an extended one-dimensional distribution. Thus, we may use the "antenna formulae," but since only the radiated effect is ask for all we need is

$$\langle \mathbf{S} \rangle = \frac{k^2 \sin^2 \theta \mu_0 c}{32\pi^2 r^2} |\mathscr{I}|^2 \hat{\mathbf{r}} \tag{6.31}$$

where

$$\mathscr{I} = \int_{-l}^{l} j_\omega(z') e^{-ikz' \cos \theta} dz' \tag{6.32}$$

so we seek a form for the current distribution $j_\omega(z')$.

It simplifies matters if we consider the cases for m an even and m an odd multiple of half wavelengths separately. As shown in Figure 6.3, for the case of m even, we will use $\sin(kz')$ and for m odd we use $\cos(kz')$.

a) even $\Rightarrow \sin(kz')$ b) odd $\Rightarrow \cos(kz')$
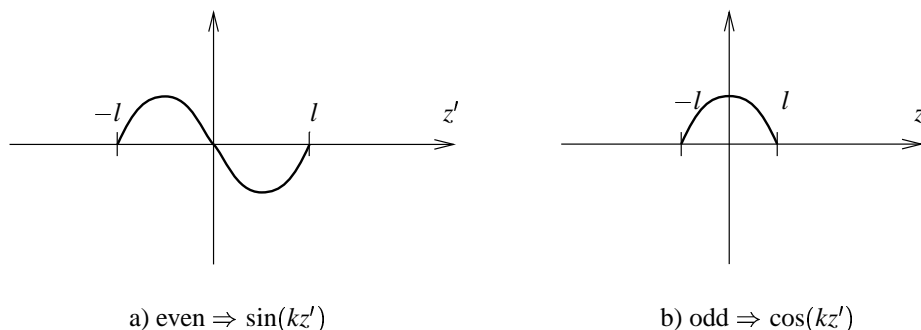
Figure 6.3. Depending on whether the length of the wire is an even (as seen in a) or odd (as seen in b) multiple of half the wave length, the current distribution is $\sin(kz')$ or $\cos(kz')$.

We perform both integrations over z' from $-l$ to l which is the total length L, which is a multiple m of $\lambda/2$. These facts give us a relation between the integration limits and k, namely

$$L = 2l = m\lambda/2 \Rightarrow l = \frac{\pi}{2}\frac{m}{k}. \tag{6.33}$$

Let us consider the odd case. The current distribution is written

$$j_\omega(z') = J_0 \cos kz' \tag{6.34}$$

and so

$$
\begin{aligned}
\mathscr{I} &= \int_{-l}^{l} J_0 \cos kz' \, e^{-ikz' \cos \theta} \, dz' = \left[\begin{array}{c} u = kz' \\ dz' = 1/k \, du \end{array} \right] \\
&= \frac{J_0}{2k} \int_{-m\pi/2}^{m\pi/2} \left(e^{iu} + e^{-iu} \right) e^{-iu \cos \theta} \, du \\
&= \frac{J_0}{2k} \left[\frac{e^{iu(1-\cos \theta)}}{i(1 - \cos \theta)} - \frac{e^{-iu(1-\cos \theta)}}{i(1 - \cos \theta)} \right]_{-m\pi/2}^{m\pi/2} \\
&= \frac{J_0}{k \sin^2 \theta} \left\{ \sin \frac{m\pi}{2}(1 - \cos \theta) + \sin \frac{m\pi}{2}(1 + \cos \theta) \right. \\
&\quad \left. + \cos \theta \left(\sin \frac{\pi}{2}(1 - \cos \theta) - \sin \frac{\pi}{2}(1 + \cos \theta) \right) \right\} \\
&= \frac{J_0}{k \sin^2 \theta} \left\{ 2 \sin(m\pi/2) \cos(m\pi/2 \cos \theta) + 2 \cos \theta \cos(m\pi/2) \sin(m\pi/2 \cos \theta) \right\} \\
&= \frac{2J_0}{k \sin^2 \theta} \cos(m\pi/2 \cos \theta) \tag{6.35}
\end{aligned}
$$

Inserting this into (6.32) gives us

$$\langle \mathbf{S} \rangle = \frac{J_0^2 \mu_0 c \cos^2(m\pi/2 \cos \theta)}{8\pi^2 r^2 \sin^2 \theta} \hat{\boldsymbol{r}} \tag{6.36}$$

Let us consider the case when m is even so the current distribution is written

$$j_\omega(z') = J_0 \sin kz'. \tag{6.37}$$

We remember the relation (6.33) which is still valid but now m is an even number. So we have that

$$\mathscr{I} = J_0 \int_{-\frac{m\pi}{2k}}^{\frac{m\pi}{2k}} e^{-ikz'\cos\theta} \sin kz' \, dz' = \left[\begin{array}{c} u = kz' \\ dz' = \frac{1}{u} du \end{array} \right]$$

$$= \frac{J_0}{2ik} \int_{-m\pi/2}^{m\pi/2} e^{-iu\cos\theta} \left(e^{iu} - e^{-iu} \right) du$$

$$= \frac{J_0}{2ik} \int_{-m\pi/2}^{m\pi/2} e^{iu(1-\cos\theta)} - e^{-iu(1+\cos\theta)} \, du$$

$$= \frac{J_0}{2ik} \left[\frac{e^{i(1-\cos\theta)\frac{m\pi}{2}} - e^{-i(1-\cos\theta)\frac{m\pi}{2}}}{i(1-\cos\theta)} - \frac{e^{i(1+\cos\theta)\frac{m\pi}{2}} - e^{-i(1+\cos\theta)\frac{m\pi}{2}}}{i(1+\cos\theta)} \right]_{-m\pi/2}^{m\pi/2}$$

$$= \frac{J_0}{ik\sin^2\theta} \left\{ (1+\cos\theta)\sin\left[(1-\cos\theta)\frac{m\pi}{2} \right] - (1-\cos\theta)\sin\left[(1+\cos\theta)\frac{m\pi}{2} \right] \right\}$$

$$= \frac{J_0}{ik\sin^2\theta} \left\{ \left(\sin\left[(1-\cos\theta)\frac{m\pi}{2} \right] - \sin\left[(1+\cos\theta)\frac{m\pi}{2} \right] \right) \right.$$
$$\left. + \cos\theta \left(\sin\left[(1-\cos\theta)\frac{m\pi}{2} \right] + \sin\left[(1+\cos\theta)\frac{m\pi}{2} \right] \right) \right\}$$

$$= \frac{2J_0}{ik\sin^2\theta} \left\{ \cos\left(\frac{m\pi}{2} \right) \sin\left(\frac{m\pi}{2}\cos\theta \right) + \cos\theta\sin\left(\frac{m\pi}{2} \right)\cos\left(\frac{m\pi}{2}\cos\theta \right) \right\}$$

$$= -i\frac{2J_0}{k\sin^2\theta} \sin\left(\frac{m\pi}{2}\cos\theta \right) \tag{6.38}$$

$$\Rightarrow |\mathscr{I}|^2 = \frac{4J_0^2}{k^2\sin^4\theta} \sin^2\left(\frac{m\pi}{2}\cos\theta \right) \tag{6.39}$$

$$\Rightarrow \langle \mathbf{S} \rangle = \frac{J_0^2 \mu_0 c}{8\pi^2 r^2} \frac{\sin^2\left(\frac{m\pi}{2}\cos\theta \right)}{\sin^2\theta} \hat{\mathbf{r}} \tag{6.40}$$

EXAMPLE 6.3   ▷TRAVELLING WAVE ANTENNA─────────────────────────────

A wire antenna of length ℓ is fed at one of its endpoints by a transmitter signal and is at its other end terminated with a resistance to ground. The termination is adjusted such that no current is reflected back on the wire. This means that the current distribution comprises travelling waves emanating from the feed point so one can assume that $j(z') = J\exp(\mathrm{i}kz')$ along the wire. Determine the angular distribution of the electromagnetic radiation from this antenna.

Solution

We need the formula

$$\langle \mathbf{S} \rangle = \frac{k^2 \sin^2 \theta}{32\pi^2 r^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} |\mathscr{I}|^2 \hat{\mathbf{r}} \tag{6.41}$$

where

$$\mathscr{I} = \int_{-L/2}^{L/2} dz' e^{-\mathrm{i}k\cos\theta z'} j(z') \tag{6.42}$$

In this case the distribution is given: $j(z') = J_0 e^{\mathrm{i}kz'}$. If we insert this distribution into (6.41), we get

$$\mathscr{I} = J_0 \int_{-L/2}^{L/2} dz' \, e^{\mathrm{i}k(1-\cos\theta)z'}$$

$$= J_0 \left[\frac{e^{\mathrm{i}k(1-\cos\theta)z'}}{\mathrm{i}k(1-\cos\theta)} \right]_{-L/2}^{L/2} = \left[\sin\alpha = \frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}\alpha} - e^{-\mathrm{i}\alpha}\right) \right]$$

$$= J_0 \frac{2}{k(1-\cos\theta)} \sin\left(\frac{kL}{2}(1-\cos\theta)\right) \tag{6.43}$$

$$|\mathscr{I}|^2 = \frac{4J_0^2}{k^2(1-\cos\theta)^2} \sin^2\left(\frac{kL}{2}(1-\cos\theta)\right) \tag{6.44}$$

Finally we have

$$\langle \mathbf{S} \rangle = \frac{J_0^2 \sin^2\theta}{8\pi^2 r^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\sin^2\left(\frac{kL}{2}(1-\cos\theta)\right)}{(1-\cos\theta)^2} \hat{\mathbf{r}} \tag{6.45}$$

───END OF EXAMPLE 6.3◁

EXAMPLE 6.4

▷MICROWAVE LINK DESIGN——————————————————————————————

Microwave links are based on direct waves, *i.e.*, propagation along the line of sight between the transmitter and receiver antennas. Reflections from the ground or a water surface may in unfavourable cases distort the transmission. Study this phenomenon using the following simple model:

The transmitter antenna T is a horizontal half-wave dipole placed a distance h_1 above the ground level. The receiving antenna R is in the main lobe from T, at a horizontal distance D from T, and at height h_2. The signal at R is considered to be composed of the direct wave and a wave reflected from the ground. The reflection is assumed to cause a phase shift π in the wave, but no loss of power. The ground is considered flat over the distance D, and $h_1, h_2 \ll D$.

1. Calculate the electric field $\mathbf{E}$ (magnitude and direction) in at R if the transmitter antenna is fed with a current j.

2. Discuss the meaning of the result.

3. Simplify the result for the case $h_1 h_2 \ll \lambda D$.
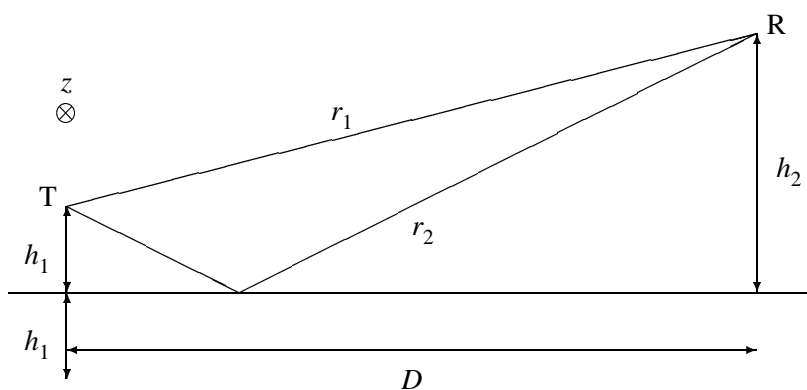
Solution

The Fourier transform of the $\mathbf{E}$ field in the far zone (radiation field) from a half-wave dipole antenna, *i.e.*, a linearly extended current distribution with length $\lambda/2$ in the z direction,

$$\mathbf{E}_{\omega,\text{rad}}(\mathbf{r}) = \frac{\mathrm{i}}{4\pi\varepsilon_0 c}\frac{e^{\mathrm{i}kr}}{r}\hat{\mathbf{r}} \times \int_{-\lambda/4}^{\lambda/4} \mathrm{d}z' e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\mathbf{j}_\omega \times \mathbf{k}$$

Since $\mathbf{k} = k\hat{\mathbf{r}}$ and we study the radiation in the maximum direction, *i.e.*, perpendicular to $\mathbf{j}_\omega(z') = J\cos(kz')\hat{\mathbf{z}}$ so that $\mathbf{k} \cdot \mathbf{x}' = 0$, this expressions simplifies to

$$\mathbf{E}_{\omega,\text{rad}}(\mathbf{r}) = \frac{ik}{4\pi\varepsilon_0 c} \frac{e^{ikr}}{r} \int_{-\lambda/4}^{\lambda/4} dz' J \cos(kz') [\hat{\mathbf{r}} \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}})]$$

$$= \frac{ik}{4\pi\varepsilon_0 c} \frac{e^{ikr}}{r} \frac{J}{k} \underbrace{[\sin(kz')]_{-\lambda/4}^{\lambda/4}}_{2} \hat{\mathbf{z}} = \frac{iJ}{2\pi\varepsilon_0 c} \frac{e^{ikr}}{r} \hat{\mathbf{z}}$$

A superposition of the direct and reflected contributions (with the distance from the transmitter T to the receiver R equal to r_1 and r_2, respectively), with due regard to the phase shift π (corresponding to a change of sign in the current), gives the Fourier transform of the total $\mathbf{E}$ field at the far zone point R:

$$\mathbf{E}_{\omega,\text{rad}}^{\text{tot}}(\mathbf{r}) = \mathbf{E}_{\omega,\text{rad}}^{\text{dir}}(\mathbf{r}) + \mathbf{E}_{\omega,\text{rad}}^{\text{refl}}(\mathbf{r}) = \frac{iJ}{2\pi\varepsilon_0 c} \left(\frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2} \right) \hat{\mathbf{z}} \qquad (6.46)$$

Since $h_1, h_2 \ll D$, Pythagoras' theorem gives

$$r_1 = \sqrt{D^2 + (h_1 - h_2)^2} \approx D + \frac{(h_1 - h_2)^2}{2D} = D + \frac{h_1^2 + h_2^2}{2D} - \frac{h_1 h_2}{D} = r - \frac{\Delta r}{2}$$

$$r_2 = \sqrt{D^2 + (h_1 + h_2)^2} \approx D + \frac{(h_1 + h_2)^2}{2D} = D + \frac{h_1^2 + h_2^2}{2D} + \frac{h_1 h_2}{D} = r + \frac{\Delta r}{2}$$

where

$$r = D + \frac{h_1^2 + h_2^2}{2D} \approx D \qquad (6.47)$$

and

$$\Delta r = \frac{2h_1 h_2}{D} \qquad (6.48)$$

is the difference in path distance. Insertion of (6.47) and (6.48) into (6.46), with $k = \omega/c = 2\pi/\lambda$, gives the Fourier transform of the field at R

$$\mathbf{E}_{\omega,\text{rad}}^{\text{tot}}(\mathbf{r}) = \frac{iJ}{2\pi\varepsilon_0 c} \frac{e^{ikr}}{D} \left(e^{-ik\Delta r/2} - e^{ik\Delta r/2} \right) \hat{\mathbf{z}} = \frac{J}{\pi\varepsilon_0 c} \frac{e^{ikr}}{D} \sin \frac{2\pi h_1 h_2}{D\lambda} \hat{\mathbf{z}}$$

from which we obtain the physical $\mathbf{E}$ field

$$\mathbf{E}_{\text{rad}}^{\text{tot}}(\mathbf{r}, t) = \text{Re} \left\{ \mathbf{E}_{\omega,\text{rad}}^{\text{tot}} e^{-i\omega t} \right\} = \frac{J}{\pi\varepsilon_0 cD} \sin \frac{2\pi h_1 h_2}{D\lambda} \cos \left[\frac{2\pi}{\lambda} \left(D + \frac{h_1^2 + h_2^2}{2D} \right) - \omega t \right] \hat{\mathbf{z}}$$

We see that the received signal at R will be extinct if

$$\frac{2\pi h_1 h_2}{D\lambda} = n\pi$$

<div style="text-align: right">———END OF EXAMPLE 6.4◁</div>

Multipole Radiation

7.1 Coverage

We look at electric dipole, magnetic dipole and electric quadrupole radiation. Multipole radiation analysis is important since it simplifies the calculation of radiation fields from time-varying field and since EM multipoles exist in many fields of physics such as astrophysics, plasma physics, atomic physics, and nuclear physics.

7.2 Formulae used

Fields in far regions from an electric dipole

$$\mathbf{B}_{\omega,\text{rad}}(\mathbf{x}) = -\frac{\omega\mu_0}{4\pi}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\,\mathbf{p}_{\omega,1}\times\mathbf{k}$$

$$\mathbf{E}_{\omega,\text{rad}}(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\left(\mathbf{p}_{\omega,1}\times\mathbf{k}\right)\times\mathbf{k}$$

Fields in the far zone from a magnetic dipole

$$\mathbf{B}_{\omega,\text{rad}}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\left(\mathbf{m}_\omega\times\mathbf{k}\right)\times\mathbf{k}$$

$$\mathbf{E}_{\omega,\text{rad}}(\mathbf{x}) = \frac{k}{4\pi\varepsilon_0 c}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\,\mathbf{m}_\omega\times\mathbf{k}$$

Field in the far zone from an electric quadrupole

$$\mathbf{B}_{\omega,\text{rad}} = \frac{i\mu_0\omega}{8\pi} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} (\mathbf{k} \cdot \mathbf{Q}) \times \mathbf{k}$$

$$\mathbf{E}_{\omega,\text{rad}} = \frac{i}{8\pi\varepsilon_0} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} [(\mathbf{k} \cdot \mathbf{Q}) \times \mathbf{k}] \times \mathbf{k}$$

7.3   Solved examples

EXAMPLE 7.1   ▷ROTATING ELECTRIC DIPOLE——————————————————————————

An electric dipole with constant electric dipole moment magnitude is located at a point in the *xy* plane and rotates with constant angular frequency.

(a)   Determine the time-dependent electromagnetic fields at large distances from the dipole.

(b)   Determine the radiated average power angular distribution and the total radiated power.

Solution

(a)   We can write the time-varying dipole momentum relative the location of the dipole as

$$\mathbf{p}(t) = p_0(\cos \omega t \, \hat{\mathbf{x}} + \sin \omega t \, \hat{\mathbf{y}}) \tag{7.1}$$

which represents a constant dipole moment p_0 times unit vector rotating with angular frequency ω. This can also be rewritten in complex form

$$\Rightarrow \mathbf{p}(t) = \text{Re} \left\{ p_0 e^{-i\omega t} \hat{\mathbf{x}} + i p_0 e^{-i\omega t} \hat{\mathbf{y}} \right\} \tag{7.2}$$

or

$$\Rightarrow \mathbf{p}(t) = (p_0 e^{-i\omega t} \hat{\mathbf{x}} + i p_0 e^{-i\omega t} \hat{\mathbf{y}}) + \text{c.c.} \tag{7.3}$$

where c.c., stands for the complex conjugate of the term opposite the $+$ sign. In what follows we use the convention that we write the dipole expression as a complex quantity but we drop the c.c. term, which is commonplace when discussing harmonic oscillation. It is easy to identify the Fourier component of the dipole moment in this case

$$\mathbf{p}_\omega(t) = p_0 \hat{\mathbf{x}} + i p_0 \hat{\mathbf{y}}. \tag{7.4}$$

We notice, in this complex space variable space, that the $\hat{\mathbf{y}}$ component has the phase factor $i = e^{i\pi/2}$ relative the $\hat{\mathbf{x}}$ component, which is due the circular rotation.

We would like to express (7.4) in spherical components rather than Cartesian components since the expressions for dipole fields in spherical components are simpler, so we transform

the base vectors $\hat{x}$ and $\hat{y}$ in the conventional fashion and get

$$\mathbf{p}_\omega(t) = p_0((\sin\theta\cos\varphi\hat{\boldsymbol{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}) \tag{7.5}$$

$$+\mathrm{i}(\sin\theta\sin\varphi\hat{\boldsymbol{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}})) = \tag{7.6}$$

$$= p_0(\hat{\boldsymbol{r}}\sin\theta(\cos\varphi + \mathrm{i}\sin\varphi) + \tag{7.7}$$

$$+\hat{\boldsymbol{\theta}}\cos\theta(\cos\varphi + \mathrm{i}\sin\varphi) + \tag{7.8}$$

$$+\hat{\boldsymbol{\varphi}}(-\sin\theta + \mathrm{i}\cos\varphi)) = \left[\cos\varphi + \mathrm{i}\sin\varphi = e^{\mathrm{i}\varphi}\right] \tag{7.9}$$

$$= p_0 e^{\mathrm{i}\varphi}(\hat{\boldsymbol{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta + \mathrm{i}\hat{\boldsymbol{\varphi}}) \tag{7.10}$$

Now that we have the Fourier component of the dipole moment expressed in spherical components we insert this into the dipole radiation fomulae:

$$\mathbf{B}_\omega = -\frac{\mu_0\omega}{4\pi}\frac{e^{\mathrm{i}kr}}{r}\mathbf{p}_\omega \times \mathbf{k} \tag{7.11}$$

$$\mathbf{E}_\omega = -\frac{1}{4\pi\varepsilon_0}\frac{e^{\mathrm{i}kr}}{r}(\mathbf{p}_\omega \times \mathbf{k}) \times \mathbf{k} \tag{7.12}$$

First we calculate

$$\mathbf{p} \times \mathbf{k} = p_0 k e^{\mathrm{i}\varphi}\begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}} \\ \sin\theta & \cos\theta & i \\ 1 & 0 & 0 \end{vmatrix} = \tag{7.13}$$

$$= p_0 k e^{\mathrm{i}\varphi}(\hat{\boldsymbol{\theta}} - \cos\theta\hat{\boldsymbol{\varphi}}) \tag{7.14}$$

from which we get

$$(\mathbf{p} \times \mathbf{k}) \times \mathbf{k} = p_0 k e^{\mathrm{i}\varphi}\begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}} \\ 0 & i & -\cos\theta \\ 1 & 0 & 0 \end{vmatrix} = \tag{7.15}$$

$$= p_0 k^2 e^{\mathrm{i}\varphi}(-\cos\theta\hat{\boldsymbol{\theta}} - i\hat{\boldsymbol{\varphi}}) \tag{7.16}$$

so finally we can write the field in space and time coordinates (remember: $\mathbf{B}(t, \mathbf{x}) = \mathrm{Re}\left\{B_\omega(\mathbf{x})e^{-\mathrm{i}\omega t}\right\}$),

$$\mathbf{B}(t, \mathbf{x}) = -\frac{\mu_0\omega}{4\pi}\mathrm{Re}\left\{\frac{e^{\mathrm{i}(kr-\omega t)}}{r}p_0 k e^{\mathrm{i}\varphi}(\hat{\boldsymbol{\theta}} - \cos\theta\varphi)\right\} = \tag{7.17}$$

$$= -\frac{\mu_0\omega p_0 k}{4\pi r}\mathrm{Re}\left\{e^{\mathrm{i}(kr-\omega t+\varphi)}e^{\mathrm{i}\varphi}(\hat{\boldsymbol{\theta}} - \cos\theta\varphi)\right\} \tag{7.18}$$

So our final expression is

$$\mathbf{B} = \frac{\mu_0\omega^2 p_0}{4\pi cr}(\sin(kr - \omega t + \varphi)\hat{\boldsymbol{\theta}} + \cos\theta\cos(kr - \omega t + \varphi)\hat{\boldsymbol{\varphi}}) \tag{7.19}$$

$$\mathbf{E} = \frac{\omega^2 p_0}{4\pi\varepsilon_0 c^2 r}(\cos\theta\cos(kr - \omega t + \varphi)\hat{\boldsymbol{\theta}} - \sin(kr - \omega t + \varphi)\hat{\boldsymbol{\varphi}}) \tag{7.20}$$

(b)   We use the formula $\langle \mathbf{S} \rangle = \frac{1}{2\mu_0} \mathbf{E} \times \mathbf{B}^*$ where the $\mathbf{E}$ and $\mathbf{B}$ fields are complex and monochromatic so

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0} \mathbf{E} \times \mathbf{B}^* = \frac{1}{2\mu_0} \frac{\mu_0 \omega}{4\pi} \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} ((\mathbf{p} \times \mathbf{k}) \times \mathbf{k}) \times (\mathbf{p}^* \times \mathbf{k}) = \tag{7.21}$$

$$= \frac{\omega}{32\pi^2\varepsilon_0 r^2} \left(\mathbf{k}(\mathbf{p} \times \mathbf{k}) \cdot (\mathbf{p}^* \times \mathbf{k}) - (\mathbf{p} \times \mathbf{k})\mathbf{k} \cdot (\mathbf{p} \times \mathbf{k}) \right) = \tag{7.22}$$

$$= \frac{\omega}{32\pi^2\varepsilon_0 r^2} |\mathbf{p} \times \mathbf{k}|^2 \hat{\mathbf{r}} = \tag{7.23}$$

$$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c r^2} (1 + \cos^2 \theta) \hat{\mathbf{r}} \tag{7.24}$$

The total power is then

$$P = \int_\Omega \langle S_r \rangle r^2 d\Omega = \tag{7.25}$$

$$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} 2\pi \int_0^\pi d\theta \sin \omega (1 + \cos^2 \theta) = \tag{7.26}$$

$$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} 2\pi \int_{-1}^1 dx (1 + x^2) = \tag{7.27}$$

$$= \frac{\mu_0 p_0^2 \omega^4}{6\pi c} \tag{7.28}$$

───────────────────────────────────────END OF EXAMPLE 7.1◁

EXAMPLE 7.2   ▷ROTATING MULTIPOLE───────────────────────────────────────

Two point charges of equal charge q are located in the xy plane at either end of the diameter of a circle of radius a. The particles rotate with a constant angular speed ω in the plane of the circle.

Determine

(a)   The Fourier components of $\mathbf{p}_1$, $\mathbf{m}$, and $\mathbf{Q}$

and

(b)   The radiation diagram when $\omega a \ll c$

Solution

$$\mathbf{r}'_1 = a(\cos \omega_0 t \hat{\mathbf{x}} + \sin \omega_0 t \hat{\mathbf{y}}) \tag{7.29}$$

$$\mathbf{r}'_2 = -\mathbf{r}'_1 \tag{7.30}$$

$$\mathbf{v}'_1 = \omega_0 a(- \sin \omega t \hat{\mathbf{x}} + \cos \omega_0 t \hat{\mathbf{y}}) \tag{7.31}$$

$$\mathbf{v}'_2 = -\mathbf{v}'_1 \tag{7.32}$$

$$\mathbf{p} = \sum_n q_n \mathbf{r}'_n = 0 \tag{7.33}$$

$$\mathbf{m} = \frac{1}{2} \sum_n \mathbf{r}'_n \times q\mathbf{v}'_n = \frac{1}{2}(\mathbf{r}'_1 \times q\mathbf{v}'_1 + (-\mathbf{r}'_1) \times q(-\mathbf{v}'_1)) = q\omega_0 a^2 \hat{\mathbf{z}} \tag{7.34}$$

$$\mathbf{m}_\omega = \hat{\mathbf{z}} q\omega_0 a^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \, d\omega = q\omega_0 a^2 \hat{\mathbf{z}} \delta(\omega) \tag{7.35}$$

$$Q_{ij} = \sum_n q_n x'_{in} x'_{jn} \tag{7.36}$$

$$Q_{xx} = q(x_1^2 + x_2^2) = 2qa^2 \cos^2 \omega_0 t = qa^2(1 + \cos 2\omega_0 t) \tag{7.37}$$

$$Q_{yy} = 2qa^2 \sin^2 \omega_0 t = qa^2(1 - \cos 2\omega_0 t) \tag{7.38}$$

$$Q_{xy} = Q_{yx} = qa^2 \sin 2\omega_0 t \tag{7.39}$$

$$Q_{iz} = Q_{zi} = 0 \tag{7.40}$$

$$Q_{xx} = qa^2(1 + e^{-i2\omega_0 t}) \tag{7.41}$$

$$Q_{yy} = qa^2(1 - e^{-i2\omega_0 t}) \tag{7.42}$$

$$Q_{xy} = Q_{yx} = qa^2 i e^{-i2\omega_0 t} \tag{7.43}$$

Fourier transform

$$Q_{xx,\omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} qa^2(1 + \cos 2\omega_0 t) e^{i\omega t} \, dt = \tag{7.44}$$

$$= qa^2(\delta(\omega) + 1/2\delta(\omega + 2\omega_0) + 1/2\delta(\omega - 2\omega_0)) \tag{7.45}$$

$$Q_{yy,\omega} = qa^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - 1/2 e^{i2\omega_0 t} - 1/2 e^{-i2\omega_0 t}) e^{i\omega t} \, dt = \tag{7.46}$$

$$= qa^2 \frac{1}{2\pi}(\delta(\omega) - \delta(\omega - 2\omega_0)) \tag{7.47}$$

$$Q_{xy,\omega} = qa^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(2\omega_0 - \omega)t} \, dt = qa^2 i \delta(\omega - 2\omega_0) \tag{7.48}$$

$$Q_\omega = qa^2 \delta(\omega) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + qa^2 \delta(\omega - 2\omega_0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.49}$$

$$\mathbf{B} = \frac{i\mu_0\omega}{8\pi}\frac{e^{ikr}}{r}(\mathbf{k}\cdot\mathbf{Q})\times\mathbf{k} \tag{7.50}$$

$$\mathbf{E} = \frac{i}{8\pi\varepsilon_0}\frac{e^{ikr}}{r}((\mathbf{k}\cdot\mathbf{Q})\times\mathbf{k})\times\mathbf{k} \tag{7.51}$$

$$\langle S\rangle = \frac{1}{2\mu_0}(\mathbf{E}\times\mathbf{B}^*) \tag{7.52}$$

EXAMPLE 7.3 ▷ATOMIC RADIATION

A transition in an atom is described as quantum matrix element of a radiation operator between the un-normalised eigenstates

$$\begin{cases} \Psi_f = re^{-r}\exp(-i\frac{E_1}{\hbar}t)\cos\theta \\ \Psi_e = e^{-r}\exp(-i\frac{E_0}{\hbar}t) \end{cases} \tag{7.53}$$

At a certain moment, the atom is therefore described by

$$\Psi = c_f\Psi_f + c_e\Psi_e \tag{7.54}$$

where c_f and c_e can be viewed as *given constants*, chosen such that Ψ becomes normalised.

According to semiclassical theory, one can interpret the magnitude squared of the wave function as a particle density function. Determine, according to this semiclassical interpretation, the power emitted by the atom via the dipole radiation which appears due to the transition between the two states.

The power from an electric dipole is given by

$$P = \frac{\mu_0\omega^4|\mathbf{p}|^2}{12\pi c} \tag{7.55}$$

Solution

The charge density is

$$\rho = q\Psi\Psi^* = q\left(|c_f\Psi_f|^2 + c_fc_e^*\Psi_e^*\Psi_f + c_f^*c_e\Psi_e\Psi_f^* + |c_e\Psi_e|^2\right) = \tag{7.56}$$

$$= q\left(|c_f|^2 r^2\exp(-2r)\cos^2\theta + |c_e|^2\exp(-2r)+ \right. \tag{7.57}$$

$$\left. + \left[c_fc_e^*\exp(-i(E_1-E_0)t) + c_f^*c_e\exp(i(E_1-E_0)t)\right]r\cos\theta\exp(-2r)\right) \tag{7.58}$$

Only the two last terms contribute because they are non-static. Via inspection we find the Fourier components for $\omega = (E_1 - E_0)/\hbar$

$$\rho_\omega(r,\theta) = qc_f^*c_e r\cos\theta\exp(-2r) \tag{7.59}$$

Consequently, the corresponding Fourier component of the dipole moment $\mathbf{p} = \int \rho_\omega \mathbf{r} \, d^3x$ is

$$\mathbf{p} = \int \left(q c_f^* c_e r \cos\theta \exp(-2r) r \hat{\mathbf{r}} \right) \, r^2 \, dr \, d\Omega = \tag{7.60}$$

$$= q c_f^* c_e \left(\int r^4 \exp(-2r) \, dr \right) \int_0^{2\pi} \int_0^{\pi} \cos\theta \sin\theta \hat{\mathbf{r}} \, d\theta \, d\phi \tag{7.61}$$

$$= q c_f^* c_e \frac{1}{2} 4! \int_0^{2\pi} \int_0^{\pi} \cos\theta \sin\theta (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \, d\theta \, d\phi = \tag{7.62}$$

$$= 12 q c_f^* c_e \left(0, 0, \int_0^{2\pi} \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \, d\phi \right) = \tag{7.63}$$

$$= 12 q c_f^* c_e \left(0, 0, 2\pi \int_0^{\pi} 1/2(\sin\theta + \sin\theta \cos 2\theta) \, d\theta \right) = \tag{7.64}$$

$$= 12\pi q c_f^* c_e \left(0, 0, 1 + 1/2 \int_0^{\pi} (\sin 3\theta - \sin\theta) \, d\theta \right) = \tag{7.65}$$

$$= 12\pi q c_f^* c_e \left(0, 0, 1 + 1/2(1/3 - 1) \right) = \tag{7.66}$$

$$= 8\pi q c_f^* c_e \left(0, 0, 1 \right) \tag{7.67}$$

$$= 8\pi q c_f^* c_e \hat{\mathbf{z}} \tag{7.68}$$

and a similar integral for the complex conjugate term. The power from the electric dipole radiation due to the transition between the two states is given by

$$P = \frac{\mu_0 \omega^4 |\mathbf{p}|^2}{12\pi c} = \frac{16\pi q^2 \mu_0 (E_1 - E_0)^4}{3 c \hbar^4} (c_f^* c_e)^2. \tag{7.69}$$

—————————————————————————END OF EXAMPLE 7.3◁

▷CLASSICAL POSITRONIUM————————————————————— EXAMPLE 7.4

Calculate the radiation from a positron-electron head-on collision and subsequent annihilation, classically by assuming that the particles travel at a constant velocity $v_0 \ll c$ up until the time the annihilate.

Solution

Background Consider a system of localised charges in motion. If we assume that we are observing at a distance $|\mathbf{x}| = R$ much greater than extension $|\mathbf{x}'|$ of the charge system and further that $v \ll c$, it can be argued that the source time t' is approximately $t - R/c$ (where $R = |\mathbf{x}|$) instead of $t - |\mathbf{x} - \mathbf{x}'|/c$. This is because, due to the first assumption

$$|\mathbf{x} - \mathbf{x}'| \approx R - \mathbf{x}' \cdot \frac{\mathbf{x}}{R} \tag{7.70}$$

Since the timescale for the system is of the order $T \sim |\mathbf{x}'|/v$ and since $v \ll c$ we have that

$$\frac{|\mathbf{x}'|}{c} \ll T \tag{7.71}$$

and this allows us to write

$$t' \approx t - R/c \tag{7.72}$$

The vector potential is this case

$$\mathbf{A} = \frac{\mu_0}{4\pi R} \int \mathbf{j}(t - R/c) \, \mathrm{d}^3 x \tag{7.73}$$

since the denominator in the integrand is now not dependent on the source coordinates. Substituting $\mathbf{j} = \rho \mathbf{v}$, we rewrite the vector potential as

$$\mathbf{A} = \frac{\mu_0}{4\pi R} \left(\Sigma q \mathbf{v} \right) \tag{7.74}$$

Observe that the summation can be written as

$$\Sigma q \mathbf{v} = \frac{d}{dt} \Sigma q \mathbf{x}' = \dot{\mathbf{p}} \tag{7.75}$$

where $\mathbf{p}$ is the electric dipole. Thus,

$$\mathbf{A} = \frac{\mu_0}{4\pi R} \dot{\mathbf{p}} \tag{7.76}$$

Deriving the EM field in the usual manner we get

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0 r c^2} \left(\left(\ddot{\mathbf{p}} \times \frac{R}{\mathbf{R}} \right) \times \frac{R}{\mathbf{R}} \right) \tag{7.77}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi r c^2} \left(\ddot{\mathbf{p}} \times \frac{R}{\mathbf{R}} \right) \tag{7.78}$$

It can be shown that the angular and spectral distribution of energy is

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}\omega \, \mathrm{d}\Omega} = \frac{\left((\ddot{\mathbf{p}})_\omega \times \mathbf{R}/R \right)^2}{2\pi c^3} \tag{7.79}$$

The Calculation   The dipole electric moment of the positronium is

$$\mathbf{p} = \begin{cases} 2q_e v_0 t \hat{z} & t \leq 0 \\ 0 & t > 0 \end{cases} \tag{7.80}$$

We take the second time derivative of this

$$\frac{\partial \mathbf{p}}{\partial t} = 2q_e v_0 \theta(-t) \hat{z} \tag{7.81}$$

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} = -2q_e v_0 \delta(t) \hat{z} \tag{7.82}$$

For the spectral and angular distributions of the radiation we have

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}\omega \, \mathrm{d}\Omega} = \frac{\left(\ddot{\mathbf{p}}_\omega \times \mathbf{R}/R \right)^2}{2\pi c^3} \tag{7.83}$$

where subscript ω denotes the Fourier transform of the dipole. Now the Fourier transform of the dipole is simply $\ddot{\mathbf{p}}_\omega = -2q v_0 \hat{z}/(2\pi)^{1/2}$, so that

$$\frac{d\mathscr{E}}{d\omega \, d\Omega} = q_e^2 v_0^2 (\hat{z} \times \hat{\mathbf{R}})^2 / \pi^2 c^3 \tag{7.84}$$

In the final result we notice that there is no dependence on ω so the spectral density is white noise.

$\rule{3.5cm}{0.4pt}$END OF EXAMPLE 7.4$\triangleleft$

Radiation from Moving Point Charges

8.1 Coverage

In our series on deriving fields from given sources we have come to the most fundamental case: the moving point charge. The fields are derived from the Liénard-Wiechert potentials. In what follows we will assume that the motion $\mathbf{x}'(t')$ is known "in advance". We find that accelerating charges radiate. We also look other mechanisms for a point charge to radiate such as Cerenkov emission.

8.2 Formulae used

According to the Formulae (F.23–26), the fields from a charge in arbitrary motion are given by

$$\mathbf{E}(t,\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 s^3}\left[\mathbf{R}_v\left(1 - \frac{v^2}{c^2}\right) + (\mathbf{x} - \mathbf{x}') \times \frac{\mathbf{R}_v \times \dot{\mathbf{v}}}{c^2}\right] \tag{8.1}$$

$$\mathbf{B}(t,\mathbf{x}) = (\mathbf{x} - \mathbf{x}') \times \frac{\mathbf{E}(t,\mathbf{x})}{c|\mathbf{x} - \mathbf{x}'|} \tag{8.2}$$

$$s = |\mathbf{x} - \mathbf{x}'| - (\mathbf{x} - \mathbf{x}') \cdot \frac{\mathbf{v}}{c} \tag{8.3}$$

$$\mathbf{R}_v = (\mathbf{x} - \mathbf{x}') - |\mathbf{x} - \mathbf{x}'| \frac{\mathbf{v}}{c} \tag{8.4}$$

$$\left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} = \frac{|\mathbf{x} - \mathbf{x}'|}{s} \tag{8.5}$$

Alternative formulae

$$\mathbf{x} - \mathbf{x}' = \mathbf{r} \tag{8.6}$$

$$\frac{\mathbf{v}}{c} = \boldsymbol{\beta}, \qquad \beta = |\boldsymbol{\beta}| = \frac{v}{c} \tag{8.7}$$

$$\mathbf{R}_v = \mathbf{x} - \mathbf{x}_0 = \mathbf{r} - r\boldsymbol{\beta} = \mathbf{r}_v = \mathbf{r}_0 \tag{8.8}$$

$$s = r - \boldsymbol{\beta} \cdot \mathbf{r} = r - \beta r \cos\theta = r(1 - \beta\cos\theta) \tag{8.9}$$

$$\left(\frac{\partial t}{\partial t'}\right)_{\mathbf{x}} = \frac{s}{r} = 1 - \beta\cos\theta \tag{8.10}$$

8.3 Solved examples

EXAMPLE 8.1   ▷POYNTING VECTOR FROM A CHARGE IN UNIFORM MOTION————————

Determine the Poynting vector for the fields from a charge q which moves with constant velocity $\mathbf{v}$. Show that no power is emitted from the charge during the motion.

Solution

In general the fields due to a single point charge may be written as

$$\mathbf{E} = \mathbf{E}_v + \mathbf{E}^{\text{rad}} \tag{8.11}$$
$$\mathbf{B} = \mathbf{B}_v + \mathbf{B}^{\text{rad}} \tag{8.12}$$

$\mathbf{E}_v$ and $\mathbf{B}_v$ are known as the velocity fields and $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ are known as acceleration fields. These can be derived from the Liénard-Wiechert potentials and result in

$$\mathbf{E}_v = \left[\frac{q}{4\pi\varepsilon_0 s^3} (\mathbf{r} - r\boldsymbol{\beta}) (1 - \beta^2) \right] \tag{8.13}$$

$$\mathbf{E}^{\text{rad}} = \left[\frac{q}{4\pi\varepsilon_0 c s^3} \left(\mathbf{r} \times \left[(\mathbf{r} - r\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right) \right] \tag{8.14}$$

and

$$\mathbf{B}_v = \left[\frac{q}{4\pi\varepsilon_0 c s^3} \boldsymbol{\beta} \times \mathbf{r} \left(1 - \beta^2 \right) \right] \tag{8.15}$$

$$\mathbf{B}_{\text{rad}} = \left[\frac{q}{4\pi\varepsilon_0 c^2 s^3} \hat{\mathbf{r}} \times \left\{ \mathbf{r} \times \left[(\mathbf{r} - r\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\} \right] \tag{8.16}$$

where $\boldsymbol{\beta} \equiv \mathbf{v}/c$ and

$$s = r - (\boldsymbol{\beta} \cdot \mathbf{r}) \tag{8.17}$$

The big outer square brackets $\left[\; \right]$ indicate that one should evaluate their content at the retarded time $t' = t - |\mathbf{x} - \mathbf{x}'(t')|/c$, where $\mathbf{x}'(t')$ is the given motion of the charged particle. It is this that makes the equations for the fields difficult to evaluate in general. It may not be difficult to get an expression for $\mathbf{x}'(t')$, but then to solve the equation for the retarded time $t' = t - |\mathbf{x} - \mathbf{x}'(t')|/c$ for $t' = t'(t, \mathbf{x})$. In this case it is not necessary to perform this transformation of variables since we are not interested in the time evolution, so we drop the brackets.

It is easy to verify that for uniform motion of the charge q, or in other words $\dot{\boldsymbol{\beta}} \equiv 0$, that $\mathbf{E}^{\text{rad}} = \mathbf{B}^{\text{rad}} \equiv \mathbf{0}$ and that

$$\mathbf{B} = \frac{1}{c} \boldsymbol{\beta} \times \mathbf{E} \tag{8.18}$$

so the Poynting vector in this case is

$$\mathbf{S} = \frac{1}{\mu_0 c} \mathbf{E} \times (\boldsymbol{\beta} \times \mathbf{E}) = \sqrt{\frac{\varepsilon_0}{\mu_0}} \mathbf{E} \times (\boldsymbol{\beta} \times \mathbf{E}) \tag{8.19}$$

Furthermore, it can be shown that

$$s = r - (\boldsymbol{\beta} \cdot \mathbf{r}) = \sqrt{r_0^2 - (\mathbf{r}_0 \times \boldsymbol{\beta})^2} \tag{8.20}$$

where we have introduced the *virtual position* vector $\mathbf{r}_0 \equiv \mathbf{r} - r\boldsymbol{\beta}$. With these last relations we may write

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} (1 - \beta^2) \frac{1}{r_0^3 (1 - \beta^2 \sin^2 \theta)^{3/2}} \mathbf{r}_0 \tag{8.21}$$

where θ is the angle between $\mathbf{r}_0$ and $\boldsymbol{\beta}$. Inserting this relation for $\mathbf{E}$ into the relation for $\mathbf{S}$ we obtain

$$\mathbf{S} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{q^2}{16\pi^2 \varepsilon_0^2} (1 - \beta^2)^2 \frac{1}{r_0^6 (1 - \beta^2 \sin^2 \theta)^3} \mathbf{r}_0 \times (\boldsymbol{\beta} \times \mathbf{r}_0) \qquad (8.22)$$

So

$$\boldsymbol{\beta} \times \mathbf{r}_0 = \beta r_0 \sin \theta \hat{\boldsymbol{\varphi}} \qquad (8.23)$$

and

$$\mathbf{r}_0 \times (\boldsymbol{\beta} \times \mathbf{r}_0) = -\beta r_0^2 \sin \theta \hat{\boldsymbol{\theta}} \qquad (8.24)$$

So finally

$$\mathbf{S} = -\frac{q^2}{16\pi^2 \varepsilon_0} \frac{v(1 - \beta^2)^2 \sin \theta}{r_0^4 (1 - \beta^2 \sin^2 \theta)^3} \hat{\boldsymbol{\theta}} \qquad (8.25)$$

And now when we integrate the Poynting vector over a spherical surface $\mathbf{A}$ with radius R which encloses the moving charge which results in

$$P = \oint \mathbf{S} \cdot d\mathbf{A} = \oint S(\hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{r}}) d\Omega = 0 \qquad (8.26)$$

since the Poynting vector is not radial, so a charge in uniform motion *in vacuo*, does not radiate energy.

<div style="text-align: right">END OF EXAMPLE 8.1◁</div>

EXAMPLE 8.2   ▷SYNCHROTRON RADIATION PERPENDICULAR TO THE ACCELERATION————

Determine the angular distribution of synchrotron radiation in the plane perpendicular to the acceleration $\dot{\mathbf{v}}$ for a charged particle moving with velocity $\mathbf{v}$.

Solution

We consider only the formulas for the radiation fields, for which the denominator is the cube of the retarded relative distance

$$s = r - \mathbf{r} \cdot \boldsymbol{\beta} = r - r\beta \cos \theta = r(1 - \beta \cos \theta) \qquad (8.27)$$

Now, we have that $\dot{\mathbf{v}} \cdot \mathbf{r} = 0$, so

$$\mathbf{r} \times (\mathbf{r}_v \times \dot{\mathbf{v}}) = \mathbf{r}_v \underbrace{(\mathbf{r} \cdot \dot{\mathbf{v}})}_{=0} - \dot{\mathbf{v}}(\mathbf{r} \cdot \mathbf{r}_v)$$

$$= -\dot{\mathbf{v}}[\mathbf{r} \cdot (\mathbf{r} - r\boldsymbol{\beta})] \qquad (8.28a)$$

$$= -\dot{\mathbf{v}}(r^2 - r^2\beta \cos \theta) \qquad (8.28b)$$

$$= -\dot{\mathbf{v}}r^2(1 - \beta \cos \theta) \qquad (8.28c)$$

$$= -\dot{\mathbf{v}}rs \qquad (8.28d)$$

where θ is the angle between the velocity and $\mathbf{r}$. So that

$$\mathbf{E}^{\text{rad}} = -\frac{\mu_0 q}{4\pi s^3} \dot{\mathbf{v}} rs = -\frac{\mu_0 q}{4\pi} \frac{\dot{v} r}{s^2} \tag{8.29}$$

The Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0 c} |\mathbf{E}|^2 \, \hat{\mathbf{r}} = \frac{\mu_0^2 q^2}{16\pi^2 \mu_0 c} \frac{\dot{v}^2 \hat{\mathbf{r}}}{r^2 (1 - \beta \cos \theta)^4} \tag{8.30}$$

Remember that the Poynting vector represents energy flux per unit time at the *field point at the time t*. The charge's energy loss must be related to the time t', *i.e.*, the time when the energy was emitted!

$$\frac{\partial U}{\partial t} = S_r r^2 \tag{8.31}$$

but

$$\begin{aligned}
\frac{\partial U}{\partial t'} &= \frac{\partial t}{\partial t'} \frac{\partial U}{\partial t} = (1 - \beta \cos \theta) \frac{\mu_0 q^2}{16\pi^2 c} \frac{\dot{v}^2}{r^2 (1 - \beta \cos \theta)^4} r^2 \\
&= \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c} \frac{1}{(1 - \beta \cos \theta)^3}
\end{aligned} \tag{8.32}$$

—————————————————————————————END OF EXAMPLE 8.2◁

▷THE LARMOR FORMULA———————————————————————————— EXAMPLE 8.3

Derive the Larmor formula by calculating the radiated power of an accelerating point charge due to electric dipole emission. Apply the Larmor formula to linear harmonic motion and circular motion.

Solution

The Larmor formula is a very useful equation for deriving the power of emission from non-relativistic accelerating charged particles. It can be derived from the radiation fields of a non-relativistic ($v \ll c$) charged particle, but it can also be seen as an electric dipole relative a co-moving coordinate system. We shall investigate the latter.

First derive the power emitted by an electric dipole. Using a time domain (non-Fourier) version of the dipole fields:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0 r c^2} (\ddot{\mathbf{p}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} \tag{8.33}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi r c} (\ddot{\mathbf{p}} \times \hat{\mathbf{r}}) \tag{8.34}$$

we find the Poynting vector to be

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

$$= \frac{1}{16\pi^2 \varepsilon_0 r^2 c^3} \left[(\ddot{\mathbf{p}} \times \hat{\boldsymbol{r}}) \times \hat{\boldsymbol{r}} \right] \times (\ddot{\mathbf{p}} \times \hat{\boldsymbol{r}}) =$$

$$= \frac{1}{16\pi^2 \varepsilon_0 r^2 c^3} \left(|\ddot{\mathbf{p}} \times \hat{\boldsymbol{r}}|^2 \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}} \cdot (\ddot{\mathbf{p}} \times \hat{\boldsymbol{r}}) (\ddot{\mathbf{p}} \times \hat{\boldsymbol{r}}) \right) =$$

$$= \frac{1}{16\pi^2 \varepsilon_0 r^2 c^3} |\ddot{\mathbf{p}} \times \hat{\boldsymbol{r}}|^2 \hat{\boldsymbol{r}} \tag{8.35}$$

So integrating for the power $P(t)$

$$P(t) = \int_0^\pi \int_0^{2\pi} \mathbf{S} \cdot \hat{\boldsymbol{r}} r^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta$$

$$= \frac{1}{16\pi^2 \varepsilon_0 c^3} \int_0^\pi \int_0^{2\pi} |\ddot{\mathbf{p}} \times \hat{\boldsymbol{r}}|^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta$$

$$= \frac{1}{16\pi^2 \varepsilon_0 c^3} \int_0^\pi \int_0^{2\pi} |\ddot{\mathbf{p}}|^2 \sin^3\theta \, \mathrm{d}\phi \, \mathrm{d}\theta$$

$$= \frac{|\ddot{\mathbf{p}}|^2}{8\pi \varepsilon_0 c^3} \int_0^\pi \sin^3\theta \, \mathrm{d}\theta$$

$$= \frac{|\ddot{\mathbf{p}}|^2}{8\pi \varepsilon_0 c^3} \frac{4}{3}$$

$$= \frac{|\ddot{\mathbf{p}}|^2}{6\pi \varepsilon_0 c^3} \tag{8.36}$$

$$\mathbf{p}(t) = q\mathbf{x}(t) \tag{8.37}$$

$$P(t) = \frac{q^2 |\ddot{\mathbf{x}}|^2}{6\pi \varepsilon_0 c^3} = \frac{q^2 |\mathbf{a}|^2}{6\pi \varepsilon_0 c^3} \tag{8.38}$$

where we have identified the acceleration $\mathbf{a}(t) \equiv \ddot{\mathbf{x}}(t)$.

Linear harmonic motion

$$x(t) = x_0 \cos(\omega_0 t), \ a(t) = \ddot{x}(t) = -\omega_0^2 x_0 \cos(\omega_0 t). \tag{8.39}$$

$$P(t) = \frac{q^2 \omega_0^4 x_0^2 \cos^2(\omega_0 t)}{6\pi \varepsilon_0 c^3} \tag{8.40}$$

$$\langle P \rangle = \frac{q^2 \omega_0^4 x_0^2}{12\pi \varepsilon_0 c^3} \tag{8.41}$$

Circular motion

$$x(t) = R_0 \cos(\omega_0 t), \ y(t) = R_0 \sin(\omega_0 t), \tag{8.42}$$

$$P(t) = \frac{q^2 \omega_0^4 R_0^2}{6\pi\varepsilon_0 c^3} \tag{8.43}$$

Validity of the Larmor formula   The Larmor formula although not covariant in form can indeed be extended such as to be valid for all inertial frames.

One point that should definitely be raised is power radiated for more than one accelerating charge. It is not so simple that one may assume that power is proportional to the number of sources N. What must be understood is whether or not the sources are radiating coherently or incoherently. For example, consider the above case of circular motion. If we had a large collection of particles such as the case with in a particle storage ring or in circular wire, the radiation is not automatically proportional to N. If they are bunched the power is proportional to N^2, this is coherent radiation. If the charges are distributed homogeneously the radiated power is 0. And if the charges are distributed evenly but fluctuate thermally then power is proportional to N, this is incoherent radiation.

——END OF EXAMPLE 8.3◁

▷VAVILOV-ČERENKOV EMISSION—————————————————————————————   EXAMPLE 8.4

Show that the potentials at time t at a point inside the Vavilov-Čerenkov cone receive contributions from exactly two positions of the charged particle.

Solution

The motion of the charge is given by $x'(t') = v(t' - t)$ so that

$$\begin{aligned} r^2 &= (x - x')^2 + y^2 = [x - v(t' - t)]^2 + y^2 \\ &= x^2 + v^2(t - t')^2 + 2vx(t - t') + y^2 \end{aligned} \tag{8.44}$$

but $r = \frac{c}{n}(t - t')$ is the retarded distance so that

$$r^2 - \frac{c^2}{n^2}(t - t')^2 = 0 \tag{8.45}$$

or, from the expression for r^2 above:

$$x^2 + v^2(t - t')^2 + 2vx(t - t') + y^2 - \frac{c^2}{n^2}(t - t')^2 = 0 \tag{8.46}$$

$$\Rightarrow \left(v^2 - \frac{c^2}{n^2}\right)(t - t')^2 + 2vx(t - t') + y^2 + x^2 = 0 \tag{8.47}$$

This is a quadratic equation in $(t - t')$. For a fixed t we would in other words have two values of t'. It remains to show that the result is physically reasonable.

$$(t - t') = \frac{-2vx \pm \sqrt{4v^2x^2 - 4(x^2 + y^2)(v^2 - c^2/n^2)}}{2(v^2 - c^2/n^2)}$$

$$= \frac{-vx \pm \sqrt{\frac{c^2}{n^2}(x^2 + y^2) - v^2y^2}}{v^2 - \frac{c^2}{n^2}} \tag{8.48}$$

$$(t - t') \in \mathbb{R} \Rightarrow \frac{c^2}{n^2}(x^2 + y^2) > v^2y^2 \tag{8.49}$$

$$\Rightarrow \frac{c^2}{n^2v^2} > \frac{y^2}{x^2 + y^2} \tag{8.50}$$

$$\Rightarrow \sin^2 \alpha_c > \sin^2 \alpha \tag{8.51}$$

$$\Rightarrow \alpha < \alpha_c \tag{8.52}$$

where α_c is the critical angle of the Čerenkov radiation; it is half of the opening angle of the shock wave of the radiation. So that, in other words, (x, y) is inside the cone! It remains to be shown that $x < 0$

$$(t - t')\left(v^2 - \frac{c^2}{n^2}\right) = \pm\sqrt{\frac{c^2}{n^2}(x^2 + y^2) - v^2y^2} - vx$$

$$= \pm vx\sqrt{\frac{c^2}{n^2v^2}\frac{(x^2 + y^2)}{x^2} - \frac{y^2}{x^2}} - vx$$

$$= vx\left(\pm\sqrt{\sin^2 \alpha_c \frac{1}{\cos^2 \alpha} - \tan^2 \alpha} - 1\right)$$

$$= vx\left(\pm\sqrt{\frac{(1 - \cos^2 \alpha_c) - (1 - \cos^2 \alpha)}{\cos^2 \alpha}} - 1\right)$$

$$= vx\left(\pm\sqrt{1 - \frac{\cos^2 \alpha_c}{\cos^2 \alpha}} - 1\right) \tag{8.53}$$

$$(t - t')\underbrace{\left(v^2 - \frac{c^2}{n^2}\right)}_{>0} = vx\underbrace{\left(\pm\sqrt{1 - \frac{\cos^2 \alpha_c}{\cos^2 \alpha}} - 1\right)}_{\leq 0} \tag{8.54}$$

——END OF EXAMPLE 8.4◁

Radiation from Accelerated Particles

9.1 Coverage

The generation of EM fields via Liénard-Wiechert potentials are considered simultaneously with the Lorentz force to give a self-consistent treatment of radiation problems. In the previous lesson we solve radiation problems from given expressions for the motion. Now we consider how charges actually move in the EM fields and thus present the *Lorentz force*. We also discuss the effect of radiation on the motion of the radiating body itself known as *radiative reaction*

9.2 Formulae used

The covariant Lorentz force

$$\frac{d(m\gamma\mathbf{v})}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{9.1}$$

9.3   Solved examples

▷MOTION OF CHARGED PARTICLES IN HOMOGENEOUS STATIC EM FIELDS————

Solve the equations of motion for a charged particle in a static homogenous electric field **E** and magnetic field **B**.

Hint: Separate the motion **v** into motion parallel and perpendicular to the magnetic field respectively.

Solution

Background on equations of motion for charged particles   As we know, the motion of a charged particle in electric and magnetic fields is given by the *Lorentz force*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{9.2}$$

This can be seen as a definition of the **E** and **B** fields and also the fundamental equation for measuring the fields. The **E** field gives the force parallel to direction of motion and **B** gives the force perpendicular to the direction of motion. In this certain sense the Lorentz force is trivial: it is simply a definition of the EM fields.

The equation for the Lorentz force is relativistically correct as it stands, as long as one interpretes

$$\mathbf{F} = \frac{d(m\gamma\mathbf{v})}{dt} \tag{9.3}$$

where m is the mass of the particle, $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$, **v** is the three-velocity, and t is the time. The fact that the Lorentz force in this form

$$\frac{d(m\gamma\mathbf{v})}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{9.4}$$

is Lorentz invariant is not immediately clear but can easily be shown. On the other hand this equation is difficult to solve for $\mathbf{v}(t)$ because γ contains **v**.

In many cases one has conditions which are non-relativistic and under such conditions it is possible to simplify (9.3) and thus also (9.4). One simply uses the fact that as $v/c \to 0$ then $\gamma \to 1$. In this case $\mathbf{F} = md\mathbf{v}/dt$, (i.e. the Newtonian force definition), so the Lorentz force becomes

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{9.5}$$

This equation is the equation of motion for non-relativistic charge particles in EM fields.

The motion   As is well known, charged particles are uniformly accelerated in a static and homogenous electric field, and in a static and homogenous magnetic field the charged particles perform circular motion. So what happens in a combined electric and magnetic fields?

The equations of motion are given in (9.5) The first step in solving this equation is to separate the motion into motion parallel with the **B** field which we will denote with $\mathbf{v}_{\parallel}$ and

motion perpendicular to **B**, which we will denote $\mathbf{v}_\perp$; so $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$ and thus

$$\frac{d\mathbf{v}_\parallel}{dt} + \frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}\mathbf{E}_\parallel + \frac{q}{m}(\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B}) \tag{9.6}$$

and since the $\parallel$ and $\perp$ are mutually orthogonal the equation separates into

$$\frac{d\mathbf{v}_\parallel}{dt} = \frac{q}{m}\mathbf{E}_\parallel \tag{9.7}$$

and for the perpendicular direction

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B}) \tag{9.8}$$

$$\Rightarrow \frac{d\mathbf{v}_\perp}{dt} - \frac{q}{m}\mathbf{v}_\perp \times \mathbf{B} = \frac{q}{m}\mathbf{E}_\perp \tag{9.9}$$

This is a first order linear ordinary differential equation in $\mathbf{v}_\perp$. It is also inhomogeneous which means that we have both a solution to the homogenous equation and solution to the inhomogeneous equation. Let us call the solution to the inhomogeneous equation ω_D and the solution to the homogenous equation ω_R. For the inhomogeneous solution, we notice that neither the inhomogeneity nor the coefficient of the zeroth order term depends on t so the solution ω_D itself cannot be time dependent. With this assumption, and that is left is $-\omega_D \times \mathbf{B} = \mathbf{E}_\perp$. This is easily solved, by taking the cross product of this equation with **B** we find

$$-(\omega_D \times \mathbf{B}) \times \mathbf{B} = \mathbf{E}_\perp \times \mathbf{B} \tag{9.10}$$

$$\Rightarrow -\underbrace{(\omega_D \cdot \mathbf{B})}_{=0}\mathbf{B} + B^2\omega_D = \mathbf{E}_\perp \times \mathbf{B} \tag{9.11}$$

$$\Rightarrow \omega_D = \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2} \tag{9.12}$$

The solution to the homogenous equation

$$\frac{d\omega_r}{dt} = \frac{q}{m}\omega_r \times \mathbf{B} \tag{9.13}$$

on the other hand, can be shown to have the form $\omega_r = \mathbf{c}_\perp e^{-i\omega t}$ where $\mathbf{c}_\perp$ is a constant vector perpendicular to **B** and satisfies $\mathbf{c}_\perp \cdot i\mathbf{c}_\perp = 0$ (where the $\cdot$ denotes scalar product defined as the inner product of the vectors), so for example $\mathbf{c}_\perp = (1, i, 0)$ if **B** is in the 3 direction.

Interpretation of the Motion Having derived the solutions for the motion of the charged particle we are now in a position to describe it in words. First of all we see that the motion consists of three separate parts. First we have the motion along the **B** field which simply is not effected by the **B** field, so that part of the **E** field which is along the **B** field accelerates the charge as if there were no **B** field. The motion perpendicular to the **B** field can further be separated into two parts. One part, which here we have denoted ω_r, repres-
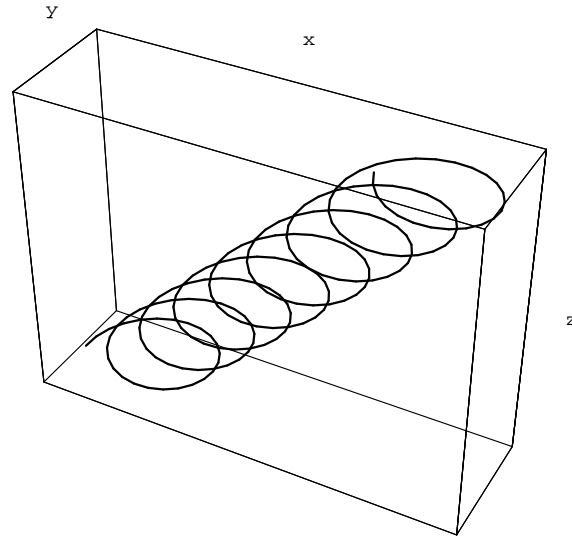
Figure 9.1. Motion of a charge in an electric and a magnetic field. Here the electric field is along the y axis and the magnetic field is along the z axis. There is a background velocity in the z axis for reasons of clarity.

ents so called gyro-harmonic rotation, i.e. the particle moves in a circular orbit, with its axis of rotation parallel with the **B** field with period of rotation $2\pi/\omega_c$ where $\omega_c = qB/m$ is known as the cyclotron frequency. This gyro-harmonic motion is the complete solution to our problem if the **E** field was not considered. The second part of the motion in the perpendicular plane however, which we here denoted ω_D involves both the **B** field and the **E** field. It is known as the drift velocity since this is the velocity of the centre of the gyro-harmonic motion.

The motion is illustrated in Figure 9.1.

END OF EXAMPLE 9.1◁

EXAMPLE 9.2　▷RADIATIVE REACTION FORCE FROM CONSERVATION OF ENERGY————————
Correct the equations of motion for a charged particle in EM fields to include the effects of the energy loss due to radiation emitted when the particle accelerates. Assume non-relativistic conditions, so radiated energy is given by Larmor formula.

Solution

Background　Among other things, Maxwell's introduction of the electric and magnetic fields was a mathematical technique to divide the work of solving the motion of charge particles in EM fields. Instead of action-at-a-distance, the fields naturally divide the prob-

lem into two parts: the generation of fields from moving charged particles and the motion of charged particles in EM fields. In a certain sense this can be seen as a division into, on the one side a cause and on the other side, an effect of EM interaction, but the formulas governing these processes are not very symmetric with respect to each other in classical electrodynamics: the generation of fields from moving point charges is determined via the Liénard-Wiechert potentials and the motion of point charged particles is determined by the Lorentz force.

Despite the lack of symmetry, this division seems successful at describing EM interaction *except for one thing: it fails to describe the self-interaction of charged point particles*. We know that the motion of charged particles is governed by the Lorentz force, but at the same time we know that acceleration of charged particles causes energy emission. These two facts have been, until now, treated separately, but taken together we realize something is missing. This can be seen for example in the case of a single charge under the influence of a mechanical force in a region of space with no EM fields except for the field from the charge itself. From the Liénard-Wiechert potentials we know that the mechanical force will cause the charge to radiate and if energy is to be conserved the emission must take its energy from the kinetic energy. Since the Lorentz force is zero and there are no other electromagnetic interactions that we know of, as of yet, we have no way of accounting for this radiative "friction". This, as of yet not mentioned force, is known as *radiative reaction* or the *radiative damping force*.

One question that comes to mind, after the above discussion, is how can so many problems be described by classical electrodynamics without considering radiative reaction? Obviously, it should have a negligible effect in most cases but what are the limiting conditions? Certainly, these should be determined by considering the conditions under which the energy emitted is of the same order as the kinetic energy of the charge. If we consider non-relativistic motion, the energy emitted by a charge accelerating at the order of **a**, under a period of duration of order T, is given by the Larmor formula and is of the order of

$$E_{\text{rad}} \sim \frac{q^2 a^2 T}{6\pi\varepsilon_0 c^3} \tag{9.14}$$

On the other hand, the acceleration bestows the charge with kinetic energy on the order of

$$E_{\text{kin}} \sim \frac{m(aT)^2}{2} \tag{9.15}$$

So if we demand that $E_{\text{rad}} \ll E_{\text{kin}}$ and we wish to neglect the radiation reaction, then this is equivalent to

$$\frac{q^2 a^2 T}{6\pi\varepsilon_0 c^3} \ll \frac{m(aT)^2}{2} \tag{9.16}$$

or

$$T \gg \frac{q^2}{3\pi\varepsilon_0 c^3} \tag{9.17}$$

and if we define the characteristic time as $\tau \equiv q^2/3\pi\varepsilon_0 c^3$ we can say that the effects of the radiative reaction are negligible in measurements made over timescales on order of $T \gg \tau$.

Accounting for radiative reaction   Having demonstrated the need for a force which accounts for radiative effects of accelerating charges we set out to determine its form. From the conservation of energy it is clear that the force we are looking for, which we will denote $\mathbf{F}_{\text{rad}}$, must satisfy

$$\int \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\, dt = -\int \frac{q^2 \dot{\mathbf{v}} \cdot \dot{\mathbf{v}}}{6\pi\varepsilon_0 c^3}\, dt \tag{9.18}$$

where have integrated the Larmor formula over time on the right hand side. Partial integration yields

$$\int \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\, dt = \int \frac{q^2 \ddot{\mathbf{v}} \cdot \mathbf{v}}{6\pi\varepsilon_0 c^3}\, dt - \left[\frac{q^2 \dot{\mathbf{v}} \cdot \mathbf{v}}{6\pi\varepsilon_0 c^3}\right] \tag{9.19}$$

if assume periodic motion then we find that

$$\int \left(\mathbf{F}_{\text{rad}} - \frac{q^2 \ddot{\mathbf{v}}}{6\pi\varepsilon_0 c^3}\right) \cdot \mathbf{v}\, dt = 0 \tag{9.20}$$

and so

$$\mathbf{F}_{\text{rad}} = \frac{q^2 \ddot{\mathbf{v}}}{6\pi\varepsilon_0 c^3} = m\tau\ddot{\mathbf{v}} \tag{9.21}$$

We now can correct the equation of motion to include the radiative reaction

$$m(\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}}) = \mathbf{F}_{\text{ext}} \tag{9.22}$$

This equation is know as *Abraham-Lorentz equation of motion*.

Unfortunately, the Abraham-Lorentz equation of motion is not without its own inherent problems.

END OF EXAMPLE 9.2◁

EXAMPLE 9.3

▷RADIATION AND PARTICLE ENERGY IN A SYNCHROTRON—————————————

An electron moves in a circular orbit in a synchrotron under the action of the homogeneous magnetic field $\mathbf{B}_0$.

(a) Calculate the energy which is lost in electromagnetic radiation per revolution if the speed v of the electron is assumed to be constant.

(b) At which particle energy does the radiated energy per revolution become equal to the particle's *total* energy, if $B_0 = 1.5\,\mathrm{T}$?

Hint:

$$\frac{d(m_0\gamma\mathbf{v})}{dt'} = \mathbf{F}, \qquad\qquad \frac{dU}{dt'} = \frac{\mu_0 q^2 \dot{\mathbf{v}}^2 \gamma^4}{6\pi c}$$

Solution

(a) Since we wish to consider synchrotron motion we use the relativistically correct equation of motion

$$\frac{d\mathbf{p}}{dt'} = q\mathbf{v} \times \mathbf{B} \tag{9.23}$$

where the right hand side is the Lorentz force with $\mathbf{E} = 0$, and $\mathbf{p}$ is the space part of the momentum 4-vector so

$$\frac{d\mathbf{p}}{dt'} = \frac{d}{dt'}(m_0\gamma\mathbf{v}) = m_0\gamma\dot{\mathbf{v}} + m_0\mathbf{v}\frac{d\gamma}{dt'} \tag{9.24}$$

where

$$\frac{d\gamma}{dt'} = \frac{d}{dt'}\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right) = -\frac{1}{2}\gamma^3\left(-\frac{d}{dt'}\frac{v^2}{c^2}\right) = \tag{9.25}$$

$$= \frac{1}{2c^2}\gamma^3\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{\gamma^3}{c^2}\mathbf{v}\cdot\dot{\mathbf{v}} \tag{9.26}$$

But $\mathbf{v}\cdot\dot{\mathbf{v}} = 0$, since we know that the electron moves in a *circular orbit*, so $\frac{d\gamma}{dt} = 0$ and thus

$$\frac{d\mathbf{p}}{dt'} = m_0\gamma\dot{\mathbf{v}} \tag{9.27}$$

This means that the power necessary to keep the particle in a circular orbit, disregarding radiation losses, is

$$\mathbf{v}\cdot\frac{d\mathbf{p}}{dt'} = m_0\gamma\left(\mathbf{v}\cdot\dot{\mathbf{v}} + \frac{\gamma^2}{c^2}v^2(\dot{\mathbf{v}}\cdot\mathbf{v})\right) = 0 \tag{9.28}$$

which comes as no surprise. However, the circular motion does emit EM radiation. To find the expression for the power loss due to radiation we need an expression for the accelera-

tion $|\dot{\mathbf{v}}|$. To this end we use what is left of (9.23), that is

$$m_0 \gamma \dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B} \tag{9.29}$$

The acceleration is found by taking the norm of this last equation and since $\mathbf{v} \parallel \mathbf{B}$ all we have is the scalar equation

$$m_0 \gamma |\dot{\mathbf{v}}| = |q|vB_0 \tag{9.30}$$

This equation is easily solved to give

$$\Rightarrow \dot{v} = \frac{|q|B_0}{m_0 \gamma}v \tag{9.31}$$

The factor $\frac{|q|B_0}{m_0 \gamma} \equiv \omega_c$ is known as the *synchrotron angular frequency* and is the relativistic value of the angular frequency for gyro-harmonic motion or *cyclotron angular frequency*.

Now we can insert this expression into the relativistic generalisation of the Larmor formula

$$\frac{dU}{dt'} = \frac{\mu_0 q^2 \dot{v}^2 \gamma^4}{6\pi c} = \frac{\mu_0 q^2 \omega_c^2 \gamma^4}{6\pi c}v^2 \tag{9.32}$$

To find the energy per revolution, we need the period of revolution which is $T = 2\pi/\omega_c$, so

$$U_{\text{rev}} = T\frac{dU}{dt'} = \frac{2\pi}{\omega_c}\frac{\mu_0 q^2 \omega_c^2 \gamma^4}{6\pi c}v^2 = \frac{\mu_0 q^2 \omega_c^2 \gamma^4}{6\pi c}v^2 = \tag{9.33}$$

$$= \frac{\mu_0 |q|^3 \gamma^3 B_0 v^2}{3cm_0} \tag{9.34}$$

(b)   The task here is to equate the total energy and the radiated energy and solve for velocity and then see what total energy that velocity is associated with.

The total energy is $E = m_0 \gamma c^2$ and the radiated energy is $U_{\text{rev}} = \mu_0 |q|^3 \gamma^3 v^2 B_0/(3cm_0)$, which when equated gives

$$\frac{c^2}{v^2}\frac{1}{\gamma^2} = \frac{\mu_0 q^3 B_0}{3cm_0^2} \tag{9.35}$$

$$\Rightarrow \frac{c^2}{v^2} - 1 = \frac{\mu_0 q^3 B_0}{3cm_0^2} \tag{9.36}$$

$$\Rightarrow \frac{v^2}{c^2} = \frac{1}{1 + \frac{\mu_0 q^3 B_0}{3cm_0^2}} \tag{9.37}$$

So, after a little algebra,

$$\gamma = \sqrt{1 + \frac{2cm_0^2}{\mu_0 |q|^3 B_0}} \tag{9.38}$$

Thus the particle energy for which the radiated energy is equal to the total particle energy is

$$E = m_0 c^2 \sqrt{1 + \frac{2cm_0^2}{\mu_0 |q|^3 B_0}} = 0.16 \, \text{TeV} \tag{9.39}$$

This is obviously the upper limit for which radiation effect can be neglected in the treatment of synchrotron motion of the particle.

———END OF EXAMPLE 9.3◁

▷RADIATION LOSS OF AN ACCELERATED CHARGED PARTICLE——————————— EXAMPLE 9.4

A charged particle, initially at rest, is accelerated by a homogeneous electric field $\mathbf{E}_0$. Show that the radiation loss is negligible, even at relativistic speeds, compared to the particle's own energy gain. Assume that under all circumstances $E_0 < 10^8$ V/m.

Solution

The relativistic equation of motion is

$$\frac{d\mathbf{p}}{dt'} = \frac{d}{dt'}(m_0 \gamma \mathbf{v}) = m_0 \gamma \left(\dot{\mathbf{v}} + \frac{\gamma^2}{c^2} \mathbf{v}(\dot{\mathbf{v}} \cdot \mathbf{v}) \right) = q\mathbf{E}_0 \tag{9.40}$$

Since $\mathbf{v} \parallel \dot{\mathbf{v}} \parallel \mathbf{E}_0$ then

$$m_0 \gamma \dot{v}(1 + \beta^2 \gamma^2) = m_0 \gamma \dot{v} \left(1 + \frac{\beta^2}{1 - \beta^2} \right)$$

$$= m_0 \gamma \dot{v} \left(\frac{1}{1 - \beta^2} \right) = m_0 \gamma \dot{v} \gamma^2 = |q| E_0 \tag{9.41}$$

$$\Rightarrow m_0 \gamma^3 \dot{v} = |q| E_0 \tag{9.42}$$

$$\Rightarrow \dot{v} = \frac{|q| E_0}{m_0 \gamma^3} \tag{9.43}$$

Having found $\dot{\mathbf{v}}$ we try to derive the radiation field generated by this motion; we have that

$$\begin{aligned}
\mathbf{E}^{\text{rad}} &= \frac{\mu_0 q}{4\pi s^3} \mathbf{r} \times (\mathbf{r}_v \times \dot{\mathbf{v}}) \\
&= \frac{\mu_0 q}{4\pi} \frac{1}{r^3 (1 - \beta \cos \theta)^3} \mathbf{r} \times \left[\left(\mathbf{r} - r\frac{\mathbf{v}}{c} \right) \times \dot{\mathbf{v}} \right] \\
&= \frac{\mu_0 q}{4\pi} \frac{1}{r^3 (1 - \beta \cos \theta)^3} \mathbf{r} \times \left(\mathbf{r} \times \dot{\mathbf{v}} - \frac{r}{c} \underbrace{\mathbf{v} \times \dot{\mathbf{v}}}_{=0} \right) \\
&= \frac{\mu_0 q}{4\pi} \frac{1}{r^3 (1 - \beta \cos \theta)^3} \mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{v}}) \\
&= \frac{\mu_0 q}{4\pi} \frac{1}{r^3 (1 - \beta \cos \theta)^3} r^2 \dot{v} \sin \theta \hat{\boldsymbol{\theta}} \\
&= \frac{\mu_0 q \dot{v}}{4\pi} \frac{\sin \theta}{r (1 - \beta \cos \theta)^3} \hat{\boldsymbol{\theta}} \tag{9.44}
\end{aligned}$$

where have used the particular geometry of the vectors involved. Then we may determine the Poynting vector, which is

$$\mathbf{S} = \frac{1}{\mu_0 c} \left| \mathbf{E}^{\mathrm{rad}} \right|^2 \hat{\mathbf{r}} = S_r \hat{\mathbf{r}} \tag{9.45}$$

The radiated energy per unit area per unit time corresponding to this is

$$\frac{\partial U}{\partial t} = S_r r^2 = \frac{\left| \mathbf{E}^{\mathrm{rad}2} \right|}{\mu_0 c} r^2 \tag{9.46}$$

but the energy radiated per unit area per unit time at the source point is

$$
\begin{aligned}
\frac{\partial U}{\partial t'} &= \frac{\partial U}{\partial t}\frac{\partial t}{\partial t'} = \frac{\partial U}{\partial t}\frac{s}{r} = \frac{rs}{\mu_0 c} \left| \mathbf{E}^{\mathrm{rad}} \right|^2 \\
&= \frac{r^2 (1 - \beta \cos\theta)}{\mu_0 c} \frac{\mu_0^2 q^2 \dot{v}^2}{16\pi^2} \frac{\sin^2\theta}{r^2 (1 - \beta \cos\theta)^6} \\
&= \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c} \frac{\sin^2\theta}{(1 - \beta \cos\theta)^5}
\end{aligned}
\tag{9.47}
$$

The total radiated energy per unit time is

$$
\begin{aligned}
\frac{\partial \tilde{U}}{\partial t'} &= \int \frac{\partial U}{\partial t'}\, \mathrm{d}\Omega = \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c} 2\pi \int_0^\pi \frac{\sin^3\theta}{(1 - \beta \cos\theta)^5}\, \mathrm{d}\theta \\
&= \left[\begin{array}{c} x = -\cos\theta \\ \mathrm{d}\theta = \frac{\mathrm{d}x}{\sin\theta} \end{array} \right] = 2\pi \frac{\mu_0 q^2 \dot{v}^2}{8\pi^2 c} \int_{-1}^1 \frac{1 - x^2}{(1 + \beta x)^5}\, \mathrm{d}x \\
&= 2\pi \frac{\mu_0 q^2 \dot{v}^2}{8\pi^2 c} \int_{-1}^1 \left(\frac{1}{(1 + \beta x)^5} - \frac{x^2}{(1 + \beta x)^5} \right) dx \\
&= 2\pi \frac{\mu_0 q^2 \dot{v}^2}{8\pi^2 c}\frac{4}{3}\frac{1}{(1 - \beta^2)^3} = 2\pi \frac{\mu_0 q^2 \dot{v}^2}{8\pi^2 c}\frac{4}{3}\gamma^6 = \frac{\mu_0 q^4 E_0^2}{6\pi m_0^2 c}
\end{aligned}
\tag{9.48}
$$

We compare this expression for the radiated energy with the total energy $E = m_0 \gamma c^2$, so

$$\frac{\mathrm{d}E}{\mathrm{d}t'} = m_0 c^2 \frac{\mathrm{d}\gamma}{\mathrm{d}t'} \tag{9.49}$$

From the equation of motion we find that

$$\frac{d}{dt'}(m_0 \gamma v) = m_0 \gamma \dot{v} + m_0 v \frac{\mathrm{d}\gamma}{dt'} = |q|E_0 \tag{9.50}$$

and, from the expression for $\dot{v}$,

$$m_0 \gamma \dot{v} = \frac{|q|E_0}{\gamma^2} \tag{9.51}$$

Hence

$$m_0 v \frac{d\gamma}{dt'} = |q|E_0 \underbrace{\left(1 - \frac{1}{\gamma^2}\right)}_{\frac{v^2}{c^2}} \tag{9.52}$$

$$\Rightarrow \frac{d\gamma}{dt'} = \frac{|q|E_0 v}{m_0 c^2} \tag{9.53}$$

and

$$\frac{dE}{dt'} = m_0 c^2 \frac{|q|E_0 v}{m_0 c^2} = |q|E_0 v \tag{9.54}$$

Finally, for $E_0 = 10^8$ V/m, and an electron for which $|q| = e$, we find that

$$\frac{d\tilde{U}}{dt'} \Big/ \frac{dE}{dt'} = \frac{\mu_0 q^4 E_0^2}{6\pi c m_0^2} \frac{1}{|q|E_0 v} = \frac{\mu_0 |e|^3 E_0}{6\pi c m_0^2 v}$$
$$= \frac{E_0}{v} \frac{(4\pi \times 10^{-7}) \cdot (1.6 \times 10^{-19})^3}{6\pi \cdot 2.998 \times 10^8 \cdot (9.11 \times 10^{-31})^2} = 1.1 \times 10^{-12} \frac{E_0}{v} \tag{9.55}$$

Which of course is a very small ratio for a relativistic electron ($v \lesssim c = 2.998 \times 10^8$ m/s) even for $E = 10^8$ V/m.

————————————————————————————END OF EXAMPLE 9.4◁